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Inverse and Direct Problems

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Differential Equations
Inverse and Direct Problems

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Preface

The meeting on *Differential Equations: Inverse and Direct Problems* was held in Cortona, June 21-25, 2004. The topics discussed by well-known specialists in the various disciplinary fields during the Meeting included, among others: differential and integrodifferential equations in Banach spaces, linear and nonlinear theory of semigroups, direct and inverse problems for regular and singular elliptic and parabolic differential and/or integrodifferential equations, blow up of solutions, elliptic equations with Wentzell boundary conditions, models in superconductivity, phase transition models, theory of attractors, Ginzburg-Landau and Schrödinger equations and, more generally, applications to partial differential and integrodifferential equations from Mathematical Physics.
The reports by the lecturers highlighted very recent, interesting and original research results in the quoted fields contributing to make the Meeting very attractive and stimulating also to younger participants.
After a lot of discussions related to the reports, some of the senior lecturers were asked by the organizers to provide a paper on their contribution or some developments of them.
The present volume is the result of all this. In this connection we want to emphasize that almost all the contributions are original and are not expositive papers of results published elsewhere. Moreover, a few of the contributions started from the discussions in Cortona and were completed in the very end of 2005.
So, we can say that the main purpose of the editors of this volume has consisted in stimulating the preparation of new research results. As a consequence, the editors want to thank in a particular way the authors that have accepted this suggestion.
Of course, we warmly thank the Italian Istituto Nazionale di Alta Matematica that made the Meeting in Cortona possible and also the Universitá degli Studi di Milano for additional support.
Finally, the editors thank the staff of Taylor & Francis for their help and useful suggestions they supplied during the preparation of this volume.

Angelo Favini and Alfredo Lorenzi

Bologna and Milan, December 2005

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Degenerate first order identification problems in Banach spaces [1]

Mohammed Al-Horani and Angelo Favini

Abstract We study a first order identification problem in a Banach space. We discuss both the nondegenerate and (mainly) the degenerate case. As a first step, suitable hypotheses on the involved closed linear operators are made in order to obtain unique solvability after reduction to a nondegenerate case; the general case is then handled with the help of new results on convolutions. Various applications to partial differential equations motivate this abstract approach.

1 Introduction

In this article we are concerned with an identification problem for first order linear systems extending the theory and methods discussed in [7] and [1]. See also [2] and [9]. Related nonsingular results were obtained in [11] under different additional conditions even in the regular case. There is a wide literature on inverse problems motivated by applied sciences. We refer to [11] for an extended list of references. Inverse problems for degenerate differential and integrodifferential equations are a new branch of research. Very recent results have been obtained in [7], [5] and [6] relative to identification problems for degenerate integrodifferential equations. Here we treat similar equations without the integral term and this allows us to lower the required regularity in time of the data by one. The singular case for infinitely differentiable semigroups and second order equations in time will be treated in some forthcoming papers.

The contents of the paper are as follows. In Section 2 we present the nonsingular case, precisely, we consider the problem

$$\begin{aligned}
&u'(t) + Au(t) = f(t)z\,, \qquad 0 \le t \le \tau\,,\\
&u(0) = u_0\,,\\
&\Phi[u(t)] = g(t)\,, \qquad 0 \le t \le \tau\,,
\end{aligned}$$

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where $-A$ generates an analytic semigroup in X, X being a Banach space, $\Phi \in X^*$, $g \in C^1([0,\tau],\mathbb{R})$, $\tau > 0$ fixed, u_0, $z \in D(A)$ and the pair $(u,f) \in C^{1+\theta}([0,\tau];X) \times C^\theta([0,\tau];\mathbb{R})$, $\theta \in (0,1)$, is to be found. Here $C^\theta([0,\tau];X)$ denotes the space of all X-valued Hölder-continuous functions on $[0,\tau]$ with exponent θ, and

$$C^{1+\theta}([0,\tau];X) = \{u \in C^1([0,\tau];X);\ u' \in C^\theta([0,\tau];X)\}.$$

In Section 3 we consider the possibly degenerate problem

$$\begin{aligned} &\frac{d}{dt}((Mu)(t)) + Lu(t) = f(t)z\,, \qquad 0 \le t \le \tau\,, \\ &(Mu)(0) = Mu_0\,, \\ &\Phi[Mu(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \end{aligned}$$

where L, M are two closed linear operators in X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, for some $\theta \in (0,1)$. In this possibly degenerate problem, M may have no bounded inverse and the pair $(u,f) \in C^\theta([0,\tau];D(L)) \times C^\theta([0,\tau];\mathbb{R})$ is to be found. This problem was solved (see [1]) when $\lambda = 0$ is a simple pole for the resolvent $(\lambda L + M)^{-1}$. Here we consider this problem under the assumption that M and L act in a reflexive Banach space X with the resolvent estimate

$$\|\lambda M(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} \le C, \qquad \operatorname{Re}\lambda \ge 0\,,$$

or the equivalent one

$$\|L(\lambda M + L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T + I)^{-1}\|_{\mathcal{L}(X)} \le C, \qquad \operatorname{Re}\lambda \ge 0\,,$$

where $T = ML^{-1}$. Reflexivity of X allows to use the representation of X as a direct sum of the null space $N(T)$ and the closure of its range $R(T)$, a consequence of the ergodic theorem (see [13], pp. 216-217). Here, a basic role is played by real interpolation space, see [12].

In Section 4 we give some examples from partial differential equations describing the range of applications of the previous abstract results.

2 The nonsingular case

Let X be a Banach space with norm $\|\cdot\|_X$ (sometimes, $\|\cdot\|$ will be used for the sake of brevity), $\tau > 0$ fixed, u_0, $z \in D(A)$, where $-A$ is the generator of an analytic semigroup in X, $\Phi \in X^*$ and $g \in C^1([0,\tau],\mathbb{R})$. We want to find a pair $(u, f) \in C^{1+\theta}([0, \tau]; X) \times C^{\theta}([0, \tau]; \mathbb{R})$, $\theta \in (0, 1)$, such that

$$u'(t) + Au(t) = f(t)z\,, \qquad 0 \le t \le \tau\,, \tag{2.1}$$

$$u(0) = u_0\,, \tag{2.2}$$

$$\Phi[u(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{2.3}$$

under the compatibility relation

$$\Phi[u_0] = g(0)\,. \tag{2.4}$$

Let us remark that the compatibility relation (2.4) follows from (2.2)-(2.3).

To solve our problem we first apply Φ to (2.1) and take equation (2.3) into account; we obtain the following equation in the unknown $f(t)$:

$$g'(t) + \Phi[Au(t)] = f(t)\Phi[z]\,. \tag{2.5}$$

Suppose the condition

$$\Phi[z] \neq 0 \tag{2.6}$$

to be satisfied. Then we can write (2.5) under the form:

$$f(t) = \frac{1}{\Phi[z]}\left\{g'(t) + \Phi[Au(t)]\right\}\,, \qquad 0 \le t \le \tau\,, \tag{2.7}$$

and the solution u of (2.1)-(2.3) is assigned by the formula

$$\begin{aligned} u(t) &= e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\,\frac{\{g'(s) + \Phi[Au(s)]\}}{\Phi[z]}\,z\,ds \\ &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[Au(s)]}{\Phi[z]}\,z\,ds + e^{-tA}u_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)z\,ds\,. \end{aligned} \tag{2.8}$$

Apply the operator A to (2.8) and obtain

$$\begin{aligned} Au(t) &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[Au(s)]}{\Phi[z]}\,Az\,ds + e^{-tA}Au_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)Az\,ds\,. \end{aligned} \tag{2.9}$$

Let $Au(t) = v(t)$; then (2.7) and (2.9) can be written, respectively, as follows:

$$f(t) = \frac{1}{\Phi[z]}\left\{g'(t) + \Phi[v(t)]\right\}\,, \qquad 0 \le t \le \tau\,, \tag{2.10}$$

$$\begin{aligned} v(t) &= \int_0^t e^{-(t-s)A}\,\frac{\Phi[v(s)]}{\Phi[z]}\,Az\,ds + e^{-tA}Au_0 \\ &\quad + \frac{1}{\Phi[z]}\int_0^t e^{-(t-s)A}g'(s)Az\,ds\,. \end{aligned} \tag{2.11}$$

Let us introduce the operator S

$$Sw(t) = \int_0^t e^{-(t-s)A} \frac{\Phi[w(s)]}{\Phi[z]} Az\, ds\,.$$

Then (2.11) can be written in the form

$$v - Sv = h \tag{2.12}$$

where

$$h(t) = e^{-tA}Au_0 + \frac{1}{\Phi[z]} \int_0^t e^{-(t-s)A} g'(s)Az\, ds\,.$$

It is easy to notice that $h \in C([0,\tau];X)$.

To prove that (2.12) has a unique solution in $C([0,\tau];X)$, it is sufficient to show that S^n is a contraction for some $n \in \mathbb{N}$. For this, we note

$$\begin{aligned}
\|Sv(t)\| &\le \frac{M\,\|\Phi\|_{X^*}}{|\Phi(z)|} \int_0^t \|v(s)\|\,\|Az\|\, ds \\
\|S^2v(t)\| &\le \frac{M\,\|\Phi\|_{X^*}}{|\Phi(z)|} \int_0^t \|Tv(s)\|\,\|Az\|\, ds \\
&\le \left(\frac{M\,\|\Phi\|_{X^*}\,\|Az\|}{|\Phi(z)|}\right)^2 \int_0^t \left(\int_0^s \|v(\sigma)\|\, d\sigma\right) ds \\
&\le \left(\frac{M\,\|\Phi\|_{X^*}\,\|Az\|}{|\Phi(z)|}\right)^2 \int_0^t (t-\sigma)\|v(\sigma)\|\, d\sigma \\
&\le \left(\frac{M\,\|\Phi\|_{X^*}\,\|Az\|}{|\Phi(z)|}\right)^2 \|v\|_\infty\, \frac{t^2}{2}\,,
\end{aligned}$$

where $\|v\|_\infty = \|v\|_{C([0,\tau];X)}$.

Proceeding by induction, we can find the estimate

$$\|S^n v(t)\| \le \left(\frac{M\,\|\Phi\|_{X^*}\|Az\|}{|\Phi(z)|}\right)^n \frac{t^n}{n!}\, \|v\|_\infty\,,$$

which implies that

$$\|S^n v\|_\infty \le \left(\frac{M\,\|\Phi\|_{X^*}\|Az\|}{|\Phi(z)|}\,\tau\right)^n \frac{1}{n!}\, \|v\|_\infty\,.$$

Consequently, S^n is a contraction for sufficiently large n. At last notice that $f(t)\,z$ is then a continuous $D(A)$-valued function on $[0,\tau]$, so that (2.1), (2.2) has in fact a unique strict solution. However, we want to discuss the maximal regularity for the solution $v = Au$, and for this we need some additional conditions. We now recall that if $-A$ generates a bounded analytic semigroup in X, then the real interpolation space $(X, D(A))_{\theta,\infty} = D_A(\theta,\infty)$ coincides with $\{x \in X; \sup_{t>0} t^{1-\theta}\|Ae^{-tA}x\| < \infty\}$, (see [3]).

Consider formula (2.11) and notice that (see [10])

$$e^{-tA}Au_0 \in C^{\theta}([0,\tau];X) \text{ if and only if } Au_0 \in D_A(\theta,\infty).$$

Moreover, if $g \in C^{1+\theta}([0,\tau];\mathbb{R})$ and $Az \in D_A(\theta,\infty)$, then

$$\int_0^t e^{-(t-s)A}g'(s)Az\,ds \in C^{\theta}([0,\tau];X)$$

and

$$\int_0^t e^{-(t-s)A}Az\,\Phi[v(s)]\,ds = \left(e^{-tA}Az * \Phi[v]\right)(t) \in C^{\theta}([0,\tau];X).$$

See [7] and [6].

Therefore, if we assume

$$Au_0,\ Az \in D_A(\theta,\infty), \tag{2.13}$$

then $v(t) \in C^{\theta}([0,\tau];X)$, i.e., $Au(t) \in C^{\theta}([0,\tau];X)$ which implies that $f(t) \in C^{\theta}([0,\tau];\mathbb{R})$. Then there exists a unique solution $(u,f) \in C^{1+\theta}([0,\tau];X) \times C^{\theta}([0,\tau];\mathbb{R})$.

We summarize our discussion in the following theorem.

THEOREM 2.1 *Let $-A$ be the generator of an analytic semigroup, $\Phi \in X^*$, $u_0,\ z \in D_A(\theta+1,\infty)$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. If $\Phi[z] \neq 0$ and (2.4) holds, then problem (2.1)-(2.3) admits a unique solution $(u,f) \in [C^{1+\theta}([0,\tau];X) \cap C^{\theta}([0,\tau];D(A))] \times C^{\theta}([0,\tau];\mathbb{R})$.*

---

3 The singular case

Consider the possibly degenerate problem

$$D_t(Mu) + Lu = f(t)z, \qquad 0 \le t \le \tau, \tag{3.1}$$

$$(Mu)(0) = Mu_0, \tag{3.2}$$

$$\Phi[Mu(t)] = g(t), \qquad 0 \le t \le \tau, \tag{3.3}$$

where L, M are two closed linear operators with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$ for $\theta \in (0,1)$. Here M may have no bounded inverse and the pair $(u,f) \in C([0,\tau];D(L)) \times C^{\theta}([0,\tau];\mathbb{R})$, with $Mu \in C^{1+\theta}([0,\tau];X)$, is to be determined so that the following compatibility condition must hold:

$$\Phi[Mu(0)] = \Phi[Mu_0] = g(0). \tag{3.4}$$

Let us assume that the pair (M, L) satisfies the estimate

$$\|\lambda M(\lambda M+L)^{-1}\|_{\mathcal{L}(X)} \leq C, \qquad \operatorname{Re}\lambda \geq 0\,, \tag{3.5}$$

or the equivalent one

$$\|L(\lambda M+L)^{-1}\|_{\mathcal{L}(X)} = \|(\lambda T+I)^{-1}\|_{\mathcal{L}(X)} \leq C, \qquad \operatorname{Re}\lambda \geq 0\,, \tag{3.6}$$

where $T = ML^{-1}$.

Various concrete examples of this relation can be found in [8]. One may note that $\lambda = 0$ is not necessarily a simple pole for $(\lambda + T)^{-1}$, $T = ML^{-1}$. Let $Lu = v$ and observe that $T = ML^{-1} \in \mathcal{L}(X)$. Then (3.1)-(3.3) can be written as

$$D_t(Tv) + v = f(t)z\,, \qquad 0 \leq t \leq \tau\,, \tag{3.7}$$

$$(Tv)(0) = Tv_0 = ML^{-1}v_0\,, \tag{3.8}$$

$$\Phi[Tv(t)] = g(t)\,, \qquad 0 \leq t \leq \tau\,, \tag{3.9}$$

where $v_0 = Lu_0$.

Since X is a reflexive Banach space and (3.5) holds, we can represent X as a direct sum (cfr. [8, p. 153], see also [13], pp. 216-217)

$$X = N(T) \oplus \overline{R(T)}$$

where $N(T)$ is the null space of T and $R(T)$ is the range of T. Let $\tilde{T} = T_{\overline{R(T)}} : \overline{R(T)} \to T_{\overline{R(T)}}$ be the restriction of T to $\overline{R(T)}$. Clearly $\tilde{T}$ is a one to one map from $\overline{R(T)}$ onto $R(T)$ ($\tilde{T}$ is an abstract potential operator in $\overline{R(T)}$. Indeed, in view of the assumptions, $-\tilde{T}^{-1}$ generates an analytic semigroup on $\overline{R(T)}$, (see [8, p. 154]).

Finally, let P be the corresponding projection onto $N(T)$ along $\overline{R(T)}$.

We can now prove the following theorem:

THEOREM 3.1 *Let L, M be two closed linear operators in the reflexive Banach space X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. Suppose the condition (3.5) to hold with (3.4), too. Then problem (3.1)-(3.3) admits a unique solution $(u, f) \in C^\theta([0,\tau]; D(L)) \times C^\theta([0,\tau];\mathbb{R})$ provided that*

$$\Phi[(I-P)z] \neq 0\,, \qquad \sup_{t>0} t^\theta \|(t\tilde{T}+1)^{-1}y_i\|_X < +\infty\,, \quad i = 1,\, 2$$

where $y_1 = (I-P)Lu_0$ and $y_2 = \tilde{T}^{-1}(I-P)z$.

Proof. Since P is the projection onto $N(T)$ along $\overline{R(T)}$, it is easy to check that problem (3.7)-(3.9) is equivalent to the couple of problems

$$D_t\tilde{T}(I-P)v + (I-P)v = f(t)(I-P)z\,, \qquad 0 \le t \le \tau\,, \tag{3.10}$$

$$\tilde{T}(I-P)v(0) = \tilde{T}(I-P)v_0\,, \tag{3.11}$$

$$\Phi[\tilde{T}(I-P)v(t)] = g(t)\,, \qquad 0 \le t \le \tau\,, \tag{3.12}$$

and

$$Pv(t) = f(t)Pz\,. \tag{3.13}$$

Let $w = \tilde{T}(I-P)v$, so that $(I-P)v = \tilde{T}^{-1}w$, and hence system (3.10)-(3.12) becomes

$$w'(t) + \tilde{T}^{-1}w = f(t)(I-P)z\,, \qquad 0 \le t \le \tau\,, \tag{3.14}$$

$$w(0) = w_0 = \tilde{T}(I-P)v_0 = Tv_0\,, \tag{3.15}$$

$$\Phi[w(t)] = g(t)\,, \qquad 0 \le t \le \tau\,. \tag{3.16}$$

Then, according to Theorem 2.1, there exists a unique solution $(w,f) \in C^{1+\theta}([0,\tau];\overline{R(T)}) \times C^{\theta}([0,\tau];\mathbb{R})$ with $\tilde{T}^{-1}w \in C^{\theta}([0,\tau];\overline{R(T)})$ to problem (3.14)-(3.16) provided that

$$\Phi[(I-P)z] \neq 0\,, \qquad (I-P)Lu_0\,,\ \tilde{T}^{-1}(I-P)z \in D_{\tilde{T}^{-1}}(\theta,\infty)\,.$$

Therefore, $(I-P)v \in C^{\theta}([0,\tau];\overline{R(T)})$, $Pv \in C^{\theta}([0,\tau];N(T))$ and hence there exists a unique solution $(u,f) \in C^{\theta}([0,\tau];D(L)) \times C^{\theta}([0,\tau];\mathbb{R})$ with $Mu \in C^{1+\theta}([0,\tau];X)$ to problem (3.1)-(3.3). □

Our next goal is to weaken the assumptions on the data in the Theorems 1 and 2. To this end we again suppose $-A$ to be the generator of an analytic semigroup in X of negative type, i.e., $\|e^{-tA}\| \le ce^{-\omega t}$, $t \ge 0$, where $c,\ \omega > 0$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, but we take $u_0 \in D_A(\theta+1;X)$, $z \in D_A(\theta_0,\infty)$, where $0 < \theta < \theta_0 < 1$. Our goal is to find a pair $(u,f) \in C^1([0,\tau];X) \times C([0,\tau];\mathbb{R})$, $Au \in C^{\theta}([0,\tau];X)$ such that equations (2.1)-(2.3) hold under the compatibility relation (2.4).

THEOREM 3.2 *Let $-A$ be a generator of an analytic semigroup in X of positive type, $0 < \theta < \theta_0 < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $u_0 \in D_A(\theta+1,\infty)$, $z \in D_A(\theta_0,\infty)$. If, in addition, (2.4), (2.6) hold, then problem (2.1)-(2.3) has a unique solution $(u,f) \in C^{\theta}([0,\tau],D(A)) \times C^{\theta}([0,\tau];\mathbb{R})$.*

Proof. Recall (see [10, p. 145]) that if $u_0 \in D(A)$, $f \in C([0,\tau];\mathbb{R})$, $z \in D_A(\theta_0,\infty)$, then problem (2.1)-(2.2) has a unique strict solution. Moreover, if $u_0 \in D_A(\theta+1;X)$, then the solution u to (2.1)-(2.2) has the maximal regularity u', $Au \in C([0,\tau];X) \cap B([0,\tau];D_A(\theta_0,\infty))$, where $B([0,\tau];Y)$ denotes the space of all bounded functions from $[0, \tau]$ into the Banach space Y. In addition $Au \in C^\theta([0, \tau]; X)$.

In order to prove our statement, we need to study suitably the properties of the function u and to use carefully some properties of the convolution operator and real interpolation spaces.

One readily sees that u satisfies

$$Au(t) = \int_0^t \frac{\Phi[Au(s)]}{\Phi[z]}\, Ae^{-(t-s)A}\, z\, ds + e^{-tA} Au_0 \\ + \frac{1}{\Phi[z]} \int_0^t A\, e^{-(t-s)A}\, z\, g'(s)\, ds$$

so that $v(t) = Au(t)$ must satisfy

$$v(t) = \int_0^t Ae^{-(t-s)A}\, z\, \frac{\Phi[v(s)]}{\Phi[z]}\, ds + e^{-tA} Au_0 \\ + \frac{1}{\Phi[z]} \int_0^t A\, e^{-(t-s)A}\, z\, g'(s)\, ds\,.$$

Let us introduce the operator $S : C([0, \tau]; X) \to C([0, \tau]; X)$ by

$$(Sw)(t) = \int_0^t Ae^{-(t-s)A}\, z\, \frac{\Phi[w(s)]}{\Phi[z]}\, ds\,.$$

Since $z \in D_A(\theta_0, \infty)$, i.e.,

$$\|Ae^{-tA} z\| \le \frac{c}{t^{1-\theta_0}}\,, \quad t > 0\,,$$

we deduce

$$\|Sw(t)\| \le c \int_0^t \|\Phi\|_{X^*}\, \|z\|_{\theta_0, \infty}\, \frac{\|w(s)\|}{(t-s)^{1-\theta_0}}\, ds\,,$$

$$\begin{aligned} \|S^2 w(t)\| &\le [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0, \infty}] \int_0^t \frac{\|Sw(s)\|}{(t-s)^{1-\theta_0}}\, ds \\ &\le [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0, \infty}]^2 \int_0^t \frac{ds}{(t-s)^{1-\theta_0}} \int_0^s \frac{\|w(\sigma)\|}{(s-\sigma)^{1-\theta_0}}\, d\sigma \\ &= [c\|\Phi\|_{X^*}\, \|z\|_{\theta_0, \infty}]^2 \int_0^t \left(\int_\sigma^t \frac{ds}{(t-s)^{1-\theta_0} (s-\sigma)^{1-\theta_0}} \right) \|w(\sigma)\|\, d\sigma \\ &= c_1^2 \left[\int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0} \eta^{1-\theta_0}} \right] (t-\sigma)^{1-2(1-\theta_0)} \|w(\sigma)\|\, d\sigma\,, \end{aligned}$$

where $c_1 = c\|\Phi\|_{X^*}\, \|z\|_{\theta_0, \infty}$, $\|\cdot\|_{D_A(\theta_0, \infty)}$ denoting the norm in $D_A(\theta_0, \infty)$.

Recall that

$$B(p, q) = \int_0^1 (1-\eta)^{p-1}\, \eta^{q-1}\, d\eta = \frac{\Gamma(p)\, \Gamma(q)}{\Gamma(p+q)}\,.$$

Then

$$\|S^3 w(t)\| \le c_1^3 \int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0}\eta^{1-\theta_0}} \int_0^1 \frac{d\eta}{(1-\eta)^{1-\theta_0}\eta^{2(1-\theta_0)-1}} \times \int_0^1 (t-\sigma)^{2-3(1-\theta_0)} \|w(\sigma)\| \, d\sigma$$

$$\le c_1^3 \, B(\theta_0,\theta_0) \, B(\theta_0, 2\theta_0) \int_0^1 (t-\sigma)^{2-3(1-\theta_0)} \|w(\sigma)\| \, d\sigma$$

$$\le c_1^3 \frac{\Gamma(\theta_0)^3}{\Gamma(3\theta_0)} \frac{t^{3\theta_0}}{3\theta_0} \|w\|_{C([0,t];X)} \, .$$

By induction, we easily verify that

$$\|S^n w(t)\| \le c_1^n \frac{\Gamma(\theta_0)^n}{\Gamma(n\theta_0)} \frac{t^{n\theta_0}}{n\theta_0} \|w\|_{C([0,t];X)} \, .$$

Since $\sqrt[n]{\Gamma(n\theta_0)} \to \infty$ as $n \to \infty$, we conclude that the operator S has spectral radius equal to 0. On the other hand, since $z \in D_A(\theta_0, \infty)$, $\theta_0 > \theta$, and $g' \in C^\theta([0,\tau];\mathbb{R})$, we deduce by [6] (Lemma 3.3) that the convolution

$$\int_0^t g'(s) A e^{-(t-s)A} \, z \, ds$$

belongs to $C^\theta([0,\tau];X)$.
Moreover, since $Au_0 \in D_A(\theta,\infty)$, $e^{-tA}Au_0 \in C^\theta([0,\tau];X)$. It follows that equation (2.12), i.e.,

$$v - Sv = h \, ,$$

with

$$h(t) = e^{-tA}Au_0 + \frac{1}{\Phi[z]} \int_0^t A e^{-(t-s)A} \, z \, g'(s) \, ds$$

has a unique solution $v \in C([0,\tau];X)$. In order to obtain more regularity for v, we use Lemma 3.3 in [6] (see also [7]) again. To this end, we introduce the following L^p-spaces related to any positive constant δ:

$$L^p_\delta((0,\tau);X) = \left\{ u : (0,\tau) \to X : \quad e^{-t\delta} u \in L^p((0,\tau);X) \right\} \, ,$$

endowed with the norms $\|u\|_{\delta,0,p} = \|e^{-t\delta}u\|_{L^p((0,\tau);X)}$. Moreover,

$$\|g\|_{\delta,\theta,\infty} = \|e^{-t\delta} g\|_{C^\theta([0,\tau];X)} \, .$$

Lemma 3.3 in [6] establishes that, in fact, if $z \in D_A(\theta_0,\infty))$, $0 < \theta < \theta_0 < 1$, then

$$\left\| \int_0^t A e^{-(t-s)A} \, z \, \Phi[v(s)] \, ds \right\|_{\delta,\theta,\infty} \le c \, \delta^{-\theta_0+\theta+1/p} \, \|\Phi[v(.)]\|_{\delta,0,p}$$

provided that $(\theta_0 - \theta)^{-1} < p$. Now,

$$\int_0^t |\Phi[v(t)]|^p \, e^{-\delta pt} \, dt \leq \|\Phi\|_{X^*}^p \, \|v\|_{L^p_\delta((0,\tau);X)}^p \leq \tau \, \|\Phi\|_{X^*}^p \, \|v\|_{\delta,\theta,\infty}^p \, .$$

Choose δ suitably large and recall that $h \in C^\theta([0,\tau];X)$. Then the norm of S as an operator from $C^\theta([0,\tau];X)$ (with norm $\|\cdot\|_{\delta,\theta,\infty}$) into itself is less than 1, so that we can deduce that the solution $v = Au$ has the regularity $C^\theta([0,\tau];X)$, as desired. □

As a consequence, Theorem 3.1 has the following improvement.

THEOREM 3.3 *Let L, M be two closed linear operators in the reflexive Banach space X with $D(L) \subseteq D(M)$, L being invertible, $\Phi \in X^*$ and $g \in C^{1+\theta}([0,\tau];\mathbb{R})$. Suppose (3.4), (3.5) to hold.*
If $0 < \theta < \theta_0 < 1$ and $\Phi[(I-P)z] \neq 0$, $\sup_{t>0} t^{\theta_0} \|(tT+1)^{-1}(I-P)z\|_X < +\infty$, $\sup_{t>0} t^\theta \|(tT+1)^{-1}(I-P)Lu_0\|_X < +\infty$, then problem (3.1)-(3.3) admits a unique solution $(u,f) \in C^\theta([0,\tau];D(L)) \times C^\theta([0,\tau];\mathbb{R})$ with $Mu \in C^{1+\theta}([0,\tau];X)$.

4 Applications

In this section we show that our abstract results can be applied to some concrete identification problems. For further examples for which the theory works we refer to [8].

Problem 1. Consider the following identification problem related to a bounded region Ω in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$

$$\begin{aligned}
&D_t u(x,t) = \sum_{i,j=1}^n D_{x_i}(a_{ij}(x) D_{x_j} u(x,t)) + f(t)v(x)\,, \quad (x,t) \in \Omega \times [0,\tau]\,,\\
&u(x,t) = 0\,, \quad \forall\, (x,t) \in \partial\Omega \times [0,\tau]\,,\\
&u(x,0) = u_0(x)\,, \quad x \in \Omega\,,\\
&\Phi[u(x,t)] = \int_\Omega \eta(x) u(x,t)\, dx = g(t)\,, \quad \forall\, t \in [0,\tau]\,,
\end{aligned}$$

where the coefficients a_{ij} enjoy the properties

$$a_{ij} \in C(\overline{\Omega})\,, \quad a_{ij} = a_{ji}\,,\ \ i,j = 1,2,...,n$$

$$\sum_{i,j=1}^n a_{ij}(x)\, \xi_i\, \xi_j \geq c_0 |\xi|^2 \quad \forall\, x \in \overline{\Omega}\,,\ \forall\, \xi \in \mathbb{R}^n\,,$$

c_0 being a positive constant. Moreover, $g \in C^1([0,\tau];\mathbb{R})$. We take

$$Au = -\sum_{i,j=1}^{n} D_{x_i}(a_{ij} D_{x_j} u)\,, \quad D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\,,$$

where $1 < p < +\infty$ is assumed. Concerning η, we suppose $\eta \in L^q(\Omega)$, where $1/p + 1/q = 1$. As it is well known, $-A$ generates an analytic semigroup in $L^p(\Omega)$ and thus we can apply Theorem 3.2 provided that $u_0 \in D_A(\theta + 1; \infty)$, i.e., $Au_0 \in D_A(\theta, \infty)$, $v \in D_A(\theta_0; \infty)$, $0 < \theta < \theta_0 < 1$. On the other hand, the interpolation spaces $D_A(\theta, \infty)$ are well characterized. Then our problem admits a unique solution

$$(u, f) \in C^\theta([0,\tau]; W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \times C^\theta([0,\tau];\mathbb{R}),$$

if $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $g(0) = \int_\Omega \eta(x)\, u_0(x)\, dx$ and $\int_\Omega \eta(x)\, v(x)\, dx \neq 0$.

Problem 2. Let Ω be a bounded region in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. Let us consider the identification problem

$$\begin{aligned}
&D_t u(x,t) = \sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x) D_{x_j} u(x,t)) + f(t) v(x)\,, \quad (x,t) \in \overline{\Omega} \times [0,\tau]\,,\\
&u(x,t) = 0\,, \quad (x,t) \in \partial\Omega \times [0,\tau]\,,\\
&u(x,0) = u_0(x)\,, \quad x \in \overline{\Omega}\,,\\
&\Phi[u(x,t)] = u(\overline{x},t) = g(t)\,, \quad t \in [0,\tau]\,,
\end{aligned}$$

where $\overline{x} \in \Omega$ is fixed, and the pair (f, u) is the unknown.
Here we take

$$X = C_0(\overline{\Omega}) = \left\{u \in C(\overline{\Omega}), u(x) = 0 \;\; \forall x \in \partial\Omega\right\},$$

endowed with the sup norm $\|u\|_X = \|u\|_\infty$.
If the coefficients a_{ij} are assumed as in Problem 1, and

$$Au = -\sum_{i,j=1}^{n} D_{x_i}(a_{ij}(x) D_{x_j} u(x))\,, \quad D(A) = \left\{u \in C_0(\overline{\Omega})\,;\, Au \in C_0(\overline{\Omega})\right\},$$

then $-A$ generates an analytic semigroup in X. The interpolation spaces $D_A(\theta; \infty)$ have no simple characterization, in view of the boundary conditions imposed to Au. Hence we notice that Theorem 3.2 applies provided that $u_0 \in D(A^2)$ and $v_0 \in D(A)$, $0 < \theta < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $u_0(\overline{x}) = g(0)$ and $v(\overline{x}) \neq 0$.

Notice that we could develop a corresponding result to Theorem 3.2 related to operators A with a nondense domain, but this is not so simple and the problem will be handled elsewhere.

Problem 3. Let us consider the following identification problem on a bounded region Ω in $\mathbb{R}$, $n \geq 1$, with a smooth boundary $\partial\Omega$:

$$D_t[m(x)u] = \Delta u + f(t)w(x), \quad (x,t) \in \Omega \times [0,\tau], \tag{4.1}$$

$$u = 0 \text{ on } \partial\Omega \times [0,\tau], \tag{4.2}$$

$$(mu)(x,0) = m(x)u_0(x), \quad x \in \Omega, \tag{4.3}$$

$$\int_\Omega \eta(x)\,(mu)(x,t)\,dx = g(t), \quad \forall t \in [0,\tau], \tag{4.4}$$

where $m \in L^\infty(\Omega)$, $\Delta : H_0^1(\Omega) :\longrightarrow H^{-1}(\Omega)$ is the Laplacian, $u_0 \in H_0^1(\Omega)$, $w \in H^{-1}(\Omega)$, $\eta \in H_0^1(\Omega)$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $0 < \theta < 1$, and the pair $(u,f) \in C^\theta([0,\tau];H_0^1(\Omega)) \times C^\theta([0,\tau];\mathbb{R})$ is the unknown. Of course, the integral in (4.4) stands for the duality between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. Theorem 3.3 applies with $X = H^{-1}(\Omega)$, see [8, p. 75]. We deduce that if $g(0) = \int_\Omega \eta(x)\,m(x)u_0(x)\,dx$, $w(x) = m(x)\zeta(x)$ for some $\zeta \in H_0^1(\Omega)$, $\int_\Omega \eta(x)\,m(x)\zeta(x)\,dx \neq 0$ and $(\Delta u_0)(x) = m(x)\zeta_1(x)$ for some $\zeta_1 \in H_0^1(\Omega)$, then problem (4.1)-(4.4) has a unique solution $(u,f) \in C^\theta([0,\tau];H_0^1(\Omega)) \times C^\theta([0,\tau];\mathbb{R})$, $mu \in C^{1+\theta}([0,\tau];H^{-1}(\Omega))$.

Problem 4. Consider the degenerate parabolic equation

$$D_t v = \Delta[a(x)v] + f(t)w(x), \quad (x,t) \in \Omega \times [0,\tau], \tag{4.5}$$

together with the initial-boundary conditions

$$a(x)v(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,\tau], \tag{4.6}$$

$$v(x,0) = v_0(x), \quad x \in \Omega, \tag{4.7}$$

and the additional information

$$\int_\Omega \eta(x)v(x,t)\,dx = g(t), \quad t \in [0,\tau]. \tag{4.8}$$

Here Ω is a bounded region in $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $a(x) \geq 0$ on $\overline{\Omega}$ and $a(x) > 0$ almost everywhere in Ω is a given function in $L^\infty(\Omega)$, $w \in H^{-1}(\Omega)$, $v_0 \in H_0^1(\Omega)$, $\eta \in H_0^1(\Omega)$, g is a real valued-function on $[0,\tau]$, at least continuous, and the pair (v,f) is the unknown. Of course, we shall see that functions w, v_0 and g need much more regularity. Call $a(x)v = u$. Then, if $m(x) = a(x)^{-1}$ and $u_0(x) = a(x)v_0(x)$ we obtain a system like (4.1)-(4.4). Let M be the multiplication operator by m from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ and let $L = -\Delta$ be endowed with Dirichlet condition, that is, $L : H_0^1(\Omega) \to H^{-1}(\Omega)$, as previously. Take $X = H^{-1}(\Omega)$. Then it is seen in [8, p. 81] that (3.5) holds if *i)* $a^{-1} \in L^1(\Omega)$, when $n = 1$,

ii) $a^{-1} \in L^r(\Omega)$ with some $r > 1$, when $n = 2$,

iii) $a^{-1} \in L^{\frac{n}{2}}(\Omega)$, when $n \geq 3$.

In order to apply Theorem 3.3 we suppose $u_0(x) = a(x)v_0(x) \in H_0^1(\Omega)$. Assumption (3.4) reads

$$\int_\Omega \eta(x)v_0(x)\,dx = \int_\Omega \eta(x)\frac{u_0(x)}{a(x)}\,dx = g(0)\,.$$

Take $g \in C^{1+\theta}([0,\tau];\mathbb{R}),\ 0 < \theta < 1$. Since $R(T) = R((1/a)\Delta^{-1})$, let $aw = \zeta \in H_0^1(\Omega)$, $a\Delta u_0 = a\Delta(av_0) = \zeta_1 \in H_0^1(\Omega)$, $\int_\Omega \eta(x)\frac{\zeta(x)}{a(x)}\,dx \neq 0$.
Then we conclude that there exists a unique pair (v, f) satisfying (4.5)-(4.8) with regularity

$$\Delta(av) \in C^\theta([0,\tau];H^{-1}(\Omega))\,, \quad v \in C^{1+\theta}([0,\tau];H^{-1}(\Omega))\,.$$

In many applications $a(x)$ is comparable with some power of the distance of x to the boundary $\partial\Omega$ and hence the assumptions depend heavily from the geometrical properties of the domain Ω. For example, if $\Omega = (-1,1)$, $a(x) = (1-x^2)^\alpha$ or $a(x) = (1-x)^\alpha(1+x)^\beta,\ 0 < \alpha, \beta < 1$ are allowed.
More generally, in $\mathbb{R}^n$, one can handle $a(x) = (1-\|x\|^2)^\alpha$ for some $\alpha > 0$ with $\Omega = \{x \in \mathbb{R}^n : \|x\| < r\}$, $r > 0$. Precisely, if $n = 2$, then $0 < \alpha < 1$, if $n \geq 3$ then $0 < \alpha < 2/n$.

Problem 5. Let us consider another degenerate parabolic equation, precisely

$$D_t v = x(1-x)D_x^2 v + f(t)w(x), \quad (x,t) \in (0,1) \times (0,\tau), \tag{4.9}$$

with the initial condition

$$v(x,0) = v_0(x), \quad x \in (0,1), \tag{4.10}$$

but with a Wentzell boundary condition (basic in probability theory and in applied sciences)

$$\lim_{x\to 0} x(1-x)D_x^2 v(x,t) = 0, \quad t \in (0,1).$$

We add the additional information:

$$\Phi[v(\cdot,t)] = v(\bar{x},t) = g(t), \quad t \in [0,\tau], \tag{4.11}$$

where $\bar{x} \in (0,1)$ is fixed. Here we take $X = H^1(0,1)$, with the norm

$$\|u\|_X^2 := \|u\|_{L^2(0,1)}^2 + \|u'\|_{L^2(0,1)}^2 + |u(0)|^2 + |u(1)|^2.$$

Introduce operator $(A, D(A))$ defined by

$$D(A) := \left\{u \in H^1(0,1);\ u'' \in L^1_{loc}(0,1) \text{ and } x(1-x)u'' \in H^1_0(0,1)\right\},$$
$$Au = -x(1-x)u'', \quad u \in D(A).$$

Then $-A$ generates an analytic semigroup in $H^1(0,1)$, see [8, pp. 249-250], [4]. So, we can apply Theorem 3.2; therefore, if $0 < \theta < \theta_0 < 1$, $g \in C^{1+\theta}([0,\tau];\mathbb{R})$, $v_0 \in D_A(\theta+1,\infty)$, $w \in D_A(\theta_0,\infty)$ (in particular, $v_0 \in D(A^2)$, $w \in D(A)$), $g(0) = v_0(\bar{x})$, $w(\bar{x}) \neq 0$, then there exists a unique pair $(v,f) \in C^\theta([0,\tau];D(A)) \times C^\theta([0,\tau];\mathbb{R})$ satisfying (4.9)–(4.11) and $D_t v \in C^\theta([0,\tau];H^1(0,1))$. Of course, general functionals Φ in the dual space $H(0,1)^*$ could be treated.

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A nonisothermal dynamical Ginzburg-Landau model of superconductivity. Existence and uniqueness theorems

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Abstract A time-dependent Ginzburg-Landau model describing superconductivity with thermal effects into account is studied. For this problem, the absolute temperature is a state variable for the superconductor. Therefore, we modify the classical time-dependent Ginzburg-Landau equations by including the temperature dependence. Finally, the existence and the uniqueness of this nonisothermal Ginzburg-Landau system is proved.

1 Introduction

There are some materials which exhibit a sharp rise in conductivity at temperatures of the order of $5^o K$ and currents started in these metals persist for a long time. This is the essence of superconductivity which was discovered by Kamerlingh Onnes in 1911 (cf.[1], [2], [5], [6], [7], [15], [16], [17]). He observed that the electrical resistance of various metals such as mercury, lead and tin disappeared completely in a small temperature range at a critical temperature T_c which is characteristic of the material. The complete disappearance of resistance is most sensitively demonstrated by experiments with persistent currents in superconducting rings.

In 1914 Kamerlingh Onnes discovered that the resistance of a superconductor could be restored to its value in the normal state by the application of a large magnetic field. About ten years later, Tuyn and Kamerlingh Onnes performed experiments on cylindrical specimens, with the axis along the direction of the applied field, and showed that the resistance increases rapidly in a very small field interval. The value H_c of H at which the jump in resistance occurs is termed threshold field. This value H_c is zero at $T = T_c$ and increases as the T is lowered below T_c.

In the first part of the paper we recall the London model of superconductivity, the traditional Ginzburg-Landau theory and the dynamical extension presented by Gor'kov and Éliashberg [11]. These models are able to describe the phase transition which occurs in a metal or alloy superconductor, when the temperature is constant, but under the critical value T_c. In these hypotheses the material will pass from the normal to the superconductor state if the magnetic field is lowered under the threshold field H_c. In this paper we present a generalization of the Ginzburg-Landau theory which considers variable both the magnetic field and the temperature. Also this model describes the phenomenon of superconductivity as a second-order phase transition. The two phases are represented in the plane $H - T$ by two regions divided by a parabola.

The second part of the paper is devoted to the proof of existence and uniqueness of the solutions of the nonisothermal Ginzburg-Landau equations. In a previous paper ([3]) we have shown the well posedness of the problem obtained by neglecting the magnetic field. In this paper, the existence and the uniqueness of the solutions of the nonisothermal Ginzburg-Landau equations are proved after formulating the problem by means of the classical state variables $(\psi, \mathbf{A}, \phi)$ together with the temperature $u = T/T_c$. The existence of the weak solutions in a bounded time interval is established by applying the Galerkin's technique. Then, by means of energy estimates we obtain the existence of global solutions in time. Finally, we prove further regularity and uniqueness of the solutions.

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2 Superconductivity and London theory

Until 1933 the magnetic properties of a superconductor were tacitly assumed to be a consequence of the property of infinite conductivity. Meissner and Ochsenfeld checked experimentally this assumption and found that such is not the case. They observed the behavior of a cylinder in an applied uniform magnetic field. When the temperature is above the critical value T_c, the sample is in the normal state and the internal magnetic field is equal to the external magnetic field. If the cylinder is cooled through T_c, the magnetic field inside the sample is expelled, showing that a superconducting material exhibits a perfect diamagnetism (Meissner effect).

The phenomenological theory of the brothers Heinz and Fritz London, developed in 1935 soon after the discovery of the Meissner effect, is based on the diamagnetic approach in that it gives a unique relation between current and magnetic field. At the same time it is closely related to the infinite conductivity approach in that the allowed current distributions represent a particular class of solutions for electron motion in the absence of scattering.

In the London theory the electrons of a superconducting material are divided in normal (as the electrons in a normal material scatter and suffer resistance to their motion) and superconducting (they cross the metal without suffering any resistance). Below the critical temperature T_c, the current consists of superconducting electrons and normal electrons. Above the critical temperature only normal electrons occur. Accordingly we write the current density as the sum of a normal and superconducting part, i.e.,

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s.$$

The normal density current $\mathbf{J}_n$ is required to satisfy Ohm's law, namely

$$\mathbf{J}_n = \sigma \mathbf{E}, \tag{2.1}$$

σ being the conductivity of normal electrons. In the London theory, the behavior of $\mathbf{J}_s$ is derived through a corpuscular scheme. Since the superconducting electrons suffer no resistance, their motion in the electric field $\mathbf{E}$ is governed by

$$m\dot{\mathbf{v}}_s = -q\mathbf{E}$$

where $m, -q, \mathbf{v}_s$ are the mass, the charge and the velocity of the superconducting electrons. Let n_s be the density of superconducting electrons so that $\mathbf{J}_s = -n_s q \mathbf{v}_s$. Multiplication by $-n_s q/m$ and the assumption that n_s is constant yield

$$\dot{\mathbf{J}}_s = \frac{n_s q^2}{m}\mathbf{E}. \tag{2.2}$$

Assume further that the superconductor is diamagnetic and that time variations are slow enough that the displacement current is negligible. Maxwell's equations become

$$\dot{\mathbf{B}} = -\nabla\times\mathbf{E}, \qquad \nabla\times\mathbf{B} = \mu_0 \mathbf{J}_s.$$

Comparison gives

$$\dot{\mathbf{B}} = -\frac{m}{n_s q^2}\nabla\times\dot{\mathbf{J}}_s$$

whence

$$\dot{\mathbf{B}} = -\alpha\nabla\times(\nabla\times\dot{\mathbf{B}})$$

where $\alpha = m/(\mu_0 n_s q^2)$. The usual identity $\nabla\times(\nabla\times) = \nabla(\nabla\cdot) - \Delta$ and the divergence-free condition of $\mathbf{B}$ give

$$\Delta\dot{\mathbf{B}} = \frac{1}{\alpha}\dot{\mathbf{B}}.$$

Appropriate initial values of $\mathbf{B}$ and $\mathbf{J}_s$ and an integration in time allow $\mathbf{B}$ and $\mathbf{J}_s$ to satisfy the equations

$$\Delta\mathbf{B} = \frac{1}{\alpha}\mathbf{B}, \tag{2.3}$$

$$\mathbf{B} = -\mu_0\alpha\nabla\times\mathbf{J}_s. \tag{2.4}$$

Equations (2.1) through (2.4) are the basic relations of the London theory.

Take the curl of (2.1) and compare with (2.4) and the induction law; we have

$$\alpha\nabla\times\mathbf{J} = -\frac{\mathbf{B}}{\mu_0} - \sigma\alpha\dot{\mathbf{B}}.$$

Similarly, take the time derivative of (2.1) and compare with (2.2) to have

$$\alpha\dot{\mathbf{J}} = \frac{1}{\mu_0}\mathbf{E} + \alpha\sigma\dot{\mathbf{E}}.$$

These equations have the advantage that only the total current density $\mathbf{J}$ occurs rather than $\mathbf{J}_n$ and $\mathbf{J}_s$. This is appreciated if the separation of the total current $\mathbf{J}$ into the two components $\mathbf{J}_n, \mathbf{J}_s$ looks somewhat artificial.

Equations (2.3) and (2.4) can be used to evaluate the magnetic induction (field) in a superconductor. If, for simplicity, $\mathbf{B}$ is allowed to depend on a single Cartesian coordinate, x say, then by (2.3) the only bounded solution as $x \geq 0$ is

$$\mathbf{B}(x) = \mathbf{B}(0)\exp(-x/\sqrt{\alpha}).$$

This result shows that, roughly, $\mathbf{B}$ penetrates in the half-space $x \geq 0$ of a distance $\sqrt{\alpha} = \sqrt{m/(\mu_0 n_s q^2)}$. That is why the quantity

$$\lambda_L = \frac{m}{\mu_0 n_s q^2}$$

is called London penetration depth. This implies that a magnetic field is exponentially screened from the interior of a sample with penetration depth λ_L, i.e., the Meissner effect.

3 Ginzburg-Landau theory

The Ginzburg-Landau theory [10] deals with the transition of a material from a normal state to a superconducting state. If a magnetic field occurs then the transition involves a latent heat which means that the transition is of the first order. If, instead, the magnetic field is zero the transition is associated with a jump of the specific heat and no latent heat (second-order transition). Landau [14] argued that a second-order transition induces a sudden change in the symmetry of the material and suggested that the symmetry can be measured by a complex-valued parameter ψ, called order parameter. The physical meaning of ψ is specified by saying that $|\psi|^2$ is the number density, n_s, of superconducting electrons. Hence $\psi = 0$ means that the material is in the normal state, $T > T_c$, while $|\psi| = 1$ corresponds to the state of a perfect superconductor ($T = 0$). There must exist a relation between ψ and the absolute temperature T and this occurs through the free energy e. Incidentally, at first Gorter and Casimir

[12] elaborated a thermodynamic potential with a real-valued order parameter. Later, Ginzburg and Landau argued that the order parameter should be complex-valued so as to make the theory gauge-invariant.

With a zero magnetic field, at constant pressure and around the critical temperature T_c the free energy e_0 is written as

$$e_0 = -a(T)|\psi|^2 + \frac{1}{2}b(T)|\psi|^4; \tag{3.1}$$

higher-order terms in $|\psi|^2$ are neglected which means that the model is valid around the critical temperature T_c for small values of $|\psi|$. If a magnetic field occurs then the free energy of the material is given by

$$\int_\Omega e(\psi, T, \mathbf{H})dv = \int_\Omega [e_0(\psi, T) + \frac{1}{2}\mu\mathbf{H}^2 + \frac{1}{2m}| - i\hbar\nabla\psi - q\mathbf{A}\psi|^2]dv \tag{3.2}$$

where $\hbar$ is Planck's constant and $\mathbf{A}$ is the vector potential associated to $\mathbf{H}$, i.e., $\mu\mathbf{H} = \nabla \times \mathbf{A}$. The free energy (3.2) turns out to be gauge-invariant. Assume that the free energy is stationary (extremum) at equilibrium. Regard T as fixed, which means that quasi-static processes are considered whereby $\mathbf{J}_s = \nabla\times\mathbf{H}$. The corresponding Euler-Lagrange equations, for the unknowns ψ and $\mathbf{A}$, are

$$\frac{1}{2m}(i\hbar\nabla + q\mathbf{A})^2\psi - a\psi + b|\psi|^2\psi = 0, \tag{3.3}$$

$$\mathbf{J}_s = -i\frac{\hbar q}{2m}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) - \frac{q^2}{m}|\psi|^2\mathbf{A}. \tag{3.4}$$

Examine the consequences of (3.3)–(3.4). The boundary condition takes the form

$$(-i\hbar\nabla\psi - q\mathbf{A}\psi) \cdot \mathbf{n}\big|_{\partial\Omega} = 0.$$

By (3.4) this implies that

$$\mathbf{J}_s \cdot \mathbf{n}\big|_{\partial\Omega} = 0.$$

Also, letting $\psi = |\psi| \exp(i\theta)$, we obtain from (3.4) that

$$\mathbf{J}_s = -\frac{\hbar q}{m}|\psi|^2\nabla\theta - \frac{q^2}{m}|\psi|^2\mathbf{A} = -\Lambda^{-1}(\frac{\hbar}{q}\nabla\theta + \mathbf{A}). \tag{3.5}$$

Hence the London equation

$$\nabla\times(\Lambda\mathbf{J}_s) = -\mathbf{B} \tag{3.6}$$

follows.

To make the theory apparently gauge-invariant, we express the free energy in terms of $\mathbf{J}_s$ rather than of $\mathbf{A}$. As shown in [7], §3.1, we have

$$|i\hbar\nabla\psi + q\mathbf{A}\psi|^2 = \hbar^2(\nabla|\psi|)^2 + |\psi|^2(\hbar\nabla\theta + q\mathbf{A})^2 = \hbar^2(\nabla|\psi|)^2 + \Lambda\mathbf{J}_s^2.$$

Hence we can write the free energy (3.2) as a functional of $f = |\psi|$ and $T, \mathbf{H}$ in the form

$$\int_\Omega e(f,T,\mathbf{H})dv = \int_\Omega \Big[-a(T)f^2 + \frac{1}{2}b(T)f^4 + \frac{1}{2}\mu\mathbf{H}^2 + \frac{\hbar^2}{2m}(\nabla f)^2 - \frac{1}{2}\Lambda\mathbf{J}_s^2\Big]dv, \tag{3.7}$$

the sign before $\Lambda\mathbf{J}_s^2$ arising from the Legendre transformation between $\mathbf{A}$ and $\mathbf{H}$. The term $\hbar^2(\nabla f)^2/2m$ represents the energy density associated with the interaction between the superconducting phase and the normal phase.

As is the case in Ginzburg-Landau theory, we restrict attention to time-independent processes where $\mathbf{J}_s = \mathbf{J} = \nabla\times\mathbf{H}$. Hence the functional (3.7) is stationary with respect to f and $\mathbf{H}$, with $\mathbf{H}\times\mathbf{n}$ fixed at the boundary $\partial\Omega$, if the Euler-Lagrange equations

$$-\frac{\hbar^2}{2m}\Delta f + \frac{m}{2q^2f^3}\mathbf{J}_s^2 - af + bf^3 = 0 \tag{3.8}$$

$$\mu\mathbf{H} = -\nabla\times\Lambda\mathbf{J}_s \tag{3.9}$$

hold together with the boundary condition

$$\nabla f\cdot\mathbf{n}\big|_{\partial\Omega} = 0.$$

Equation (3.9) coincides with (3.6) and hence with the Ginzburg-Landau equation (3.5) or (3.4). Also, equation (3.8) reduces to equation (3.3) when the phase θ of ψ is chosen to be zero as is the case for the system (3.8), (3.9).

Since the vector potential $\mathbf{A}$ is a nonmeasurable quantity, equation (3.9) may seem more convenient than (3.4) as long as the relation $\nabla\times\Lambda\mathbf{J}_s = -\mathbf{B}$ may be preferable to (3.5).

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4 Quasi-steady model

Starting from the BCS theory of superconductivity, Schmid ([18]) and Gor'kov & Éliashberg ([11]) have elaborated a generalization to the dynamic case of the Ginzburg-Landau theory within the approximation that the temperature T is near the transition value T_c. They consider the variables ψ, $\mathbf{A}$ and the electrical potential ϕ which, together with the vector potential $\mathbf{A}$, is subject to the equations

$$\nabla\times\mathbf{A} = \mathbf{B}\,, \qquad \mathbf{E} = -\dot{\mathbf{A}} + \nabla\phi. \tag{4.1}$$

By adhering to [9] we complete the quasi-steady model of superconductivity through the equations

$$\begin{aligned} \gamma(\dot{\psi} - i\frac{q}{\hbar}\phi\psi) &= -\frac{1}{2m}(i\hbar\nabla + q\mathbf{A})^2 + a\psi - b|\psi|^2\psi, \\ \sigma(\dot{\mathbf{A}} - \nabla\phi) &= -\nabla\times\nabla\times\mathbf{A} + \mathbf{J}_s, \\ \mathbf{J}_s &= -\frac{i\hbar q}{2m}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}) - \frac{q^2}{m}|\psi|^2\mathbf{A}, \end{aligned} \tag{4.2}$$

where γ is an appropriate relaxation coefficient.

The system of equations must be invariant under a gauge transformation

$$(\psi, \mathbf{A}, \phi) \longleftrightarrow (\psi e^{i(q/\hbar)\chi}, \mathbf{A} + \nabla\chi, \phi + \dot{\chi})$$

where the gauge χ is an arbitrary smooth function of (x, t). Among the possible gauges we mention the London gauge

$$\nabla\cdot\mathbf{A} = 0\,, \qquad \mathbf{A}\cdot\mathbf{n}|_{\partial\Omega} = 0\,,$$

the Lorentz gauge

$$\nabla\cdot\mathbf{A} = -\phi$$

and the zero-electrical potential gauge $\phi = 0$. Reference [13] investigates these gauges and shows that the condition $\phi = 0$ is incompatible with the London gauge $\nabla\cdot\mathbf{A} = 0$.

The system (4.2) is associated with the initial conditions

$$\psi(x, 0) = \psi_0(x)\ , \qquad \mathbf{A}(x, 0) = \mathbf{A}_0(x). \tag{4.3}$$

Equation $(4.2)_2$ follows from the Maxwell equation

$$\nabla\times\mathbf{H} = \mathbf{J}_s + \mathbf{J}_n + \varepsilon\dot{\mathbf{E}}$$

by disregarding the derivative $\dot{\mathbf{E}}$ and letting $\mathbf{J}_n = \sigma\mathbf{E}$. That is why the problem (4.2) is called quasi-steady.

Moreover, by letting $\psi = f\exp(i\theta)$, from $(4.2)_1$, we deduce the evolution equation for the variable f. In terms of the observable variables $f, \mathbf{p}_s, \mathbf{H}, \mathbf{E}$, the system (4.2) can be written in the form

$$\gamma\dot{f} = \frac{\hbar^2}{2m}\nabla^2 f - \frac{q^2}{2m}\mathbf{p}_s^2 f + af - bf^3 \tag{4.4}$$

$$\nabla\times\mathbf{p}_s = -\mu\mathbf{H} \tag{4.5}$$

$$\nabla\times\mathbf{H} = \Lambda^{-1}(f)\mathbf{p}_s + \sigma\mathbf{E} \tag{4.6}$$

$$\nabla\times\mathbf{E} = -\mu\dot{\mathbf{H}} \tag{4.7}$$

along with the boundary conditions

$$\nabla f \cdot \mathbf{n}|_{\partial\Omega} = 0, \qquad \mathbf{H} \times \mathbf{n}|_{\partial\Omega} = g\,, \qquad \mathbf{p}_s \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{4.8}$$

and the initial conditions

$$f(x,0) = f_0(x)\,, \qquad \mathbf{H}(x,0) = \mathbf{H}_0(x). \tag{4.9}$$

Observe that by (4.5) and (4.7) we have

$$\dot{\mathbf{p}}_s = \frac{m}{q}\dot{\mathbf{v}}_s = \mathbf{E} - \nabla\phi_s.$$

This result can be viewed as the Euler equation for a nonviscous electronic liquid (see [15]), where the scalar function ϕ_s represents the thermodynamic potential per electron. The previous relation allows the quasi-steady problem (4.4)–(4.7) to be written as

$$\gamma\dot{f} = \frac{\hbar^2}{2m}\Delta f - \frac{q^2}{2m}\mathbf{p}_s^2 f + af - bf^3,$$

$$\frac{1}{\mu}\nabla\times\nabla\times\mathbf{p}_s = -\Lambda^{-1}(f)\mathbf{p}_s - \sigma\dot{\mathbf{p}}_s - \sigma\nabla\phi_s.$$

Moreover (4.6) provides

$$\nabla\cdot(\Lambda^{-1}(f)\mathbf{p}_s) = \nabla\cdot\mathbf{J}_s = -\nabla\cdot(\sigma\mathbf{E}) = -\sigma\rho. \tag{4.10}$$

In the theory of Gor'kov and Éliashberg [11], which is based on the system (4.2), the function ϕ_s is assumed to depend on f and on the total electron density ρ in the form

$$\phi_s = \Lambda(f)\rho. \tag{4.11}$$

The comparison of (4.10) and (4.11) gives

$$\nabla\cdot(\Lambda^{-1}(f)\mathbf{p}_s) = -\sigma\Lambda^{-1}(f)\phi_s. \tag{4.12}$$

5 Phase transition in superconductivity with thermal effects

We present a generalization of the model which describes the phase transition in superconductivity without neglecting thermal effects. The main assumption is that the phase transition is of second order and that the effects due to the variation of the temperature are like the ones shown by varying the magnetic field. In this sense the temperature T can be considered as the dual variable of the magnetic field $\mathbf{H}$.

In order to justify the model here examined, we consider the expression of Gauss free energy in terms of the variables $(\psi, T, \mathbf{A})$

$$\begin{aligned}&\mathcal{E}(\psi, u, \mathbf{A})\\&= \int_\Omega \left[-a(T)|\psi|^2 + \frac{1}{2} b(T)|\psi|^4 + \frac{1}{2\mu}|\nabla \times \mathbf{A}|^2 + \frac{1}{2m}|i\hbar\nabla\psi + q\mathbf{A}\psi|^2 \right] dv.\end{aligned}$$

Following [8] and [19] we consider the linear approximation of $a(T)$ in a neighborhood of the critical temperature, namely

$$a(T) = -a_0 \left(\frac{T}{T_c} - 1 \right) = -a_0(u-1),$$

where $u = \frac{T}{T_c} > 0$. Finally, we suppose constant the coefficient $b(T)$. By means of the temperature u, the critical value u_c is now given by $u_c = 1$, while the domain of definition is $\mathbb{R}^+$.

Under these hypotheses the free energy takes the following form

$$\begin{aligned}&\mathcal{E}(\psi, u, \mathbf{A})\\&= \int_\Omega \left[a_0(u-1)|\psi|^2 + \frac{b_0}{2}|\psi|^4 + \frac{1}{2\mu}|\nabla \times \mathbf{A}|^2 + \frac{1}{2m}|i\hbar\nabla\psi + q\mathbf{A}\psi|^2 \right] dv\end{aligned} \tag{5.1}$$

which as a function of f, u, H can be written as

$$E(f, u, \mathbf{H}) = \int_\Omega \left[a_0(u-1)f^2 + \frac{b_0}{2}f^4 + \frac{\mu}{2}\mathbf{H}^2 + \frac{\hbar^2}{2m}|\nabla f|^2 + \frac{1}{2}\Lambda(f)|\nabla \times \mathbf{H}|^2 \right] dv.$$

When we use the representation (5.1) as free energy with $a_0 = b_0 = 1$, then the first Gor'kov Éliashberg equation takes the dimensionless form

$$\gamma \dot{f} = \frac{1}{\kappa^2}\triangle f - (f^2 - 1 + u)f - f|\mathbf{A} - \frac{1}{\kappa}\nabla\theta|^2 \tag{5.2}$$

$$\triangle\phi + \gamma f^2(\dot{\theta} - \kappa\phi) = 0 \tag{5.3}$$

where $\kappa > 0$ is the Ginzburg-Landau parameter. From (5.1) or (5.2) it is possible to retrieve the phase diagram, which separates the normal from superconductor zone. This relation is represented by a parabola in the $H - T$ plane (see [1]), which can be approximated considering the points for which the coefficient of f is zero. Namely, the points such that

$$-1 + u + \left| \mathbf{A} - \frac{1}{\kappa}\nabla\theta \right|^2 = 0\,.$$

The temperature effect will be supposed negligible on the first Maxwell equation, which we write in the London gauge $(\nabla \cdot \mathbf{A} = 0)$

$$\dot{\mathbf{A}} - \nabla\phi + \nabla \times \nabla \times \mathbf{A} + f^2(\mathbf{A} - \frac{1}{\kappa}\nabla\theta) = 0. \tag{5.4}$$

Finally, we need to consider the heat equation, which must be related to the equation (5.2) in order to have a thermodynamic compatibility. Hence let us consider the first law of thermodynamics or heat equation

$$\alpha u u_t - u f f_t = k \nabla \cdot u \nabla u \tag{5.5}$$

where α and k are two positive scalar constants. From (5.5), under the hypothesis of small perturbations for $|\nabla u|^2$, we obtain the entropy equation

$$\alpha u_t - f f_t = k \triangle u \tag{5.6}$$

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6 Existence and uniqueness of the solutions

In this section we prove the existence and the uniqueness of the solutions of the nonisothermal time dependent Ginzburg-Landau equations. To this purpose we write the system (4.2) in dimensionless form and the equation (5.6) by means of the complex variable ψ. Therefore we obtain

$$\gamma(\psi_t - i\kappa\phi\psi) - \frac{1}{\kappa^2}\triangle\psi + \frac{2i}{\kappa}\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi + \psi\big(|\psi|^2 - 1 + u\big) = 0\,, \tag{6.1}$$

$$\mathbf{A}_t - \nabla\phi + \nabla\times\nabla\times\mathbf{A} - \frac{i}{2\kappa}\big(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\big) + |\psi|^2\mathbf{A} = 0\,, \tag{6.2}$$

$$\alpha u_t - k\triangle u - \frac{1}{2}\big(\psi\bar{\psi}_t + \bar{\psi}\psi_t\big) = 0\,. \tag{6.3}$$

The problem is completed by the boundary conditions

$$\nabla\psi\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad (\nabla\times\mathbf{A})\times\mathbf{n}|_{\partial\Omega} = \mathbf{H}_{ex}\times\mathbf{n}, \quad \nabla\phi\cdot\mathbf{n}|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = \widetilde{u}, \tag{6.4}$$

where $\mathbf{H}_{ex}$ is the external magnetic field, and the initial data

$$\psi(x,0) = \psi_0(x)\,, \qquad \mathbf{A}(x,0) = \mathbf{A}_0(x)\,, \qquad u(x,0) = u_0(x)\,. \tag{6.5}$$

In order to deal with homogeneous boundary conditions we introduce the new variables $\widehat{u} = u - \widetilde{u}$ and $\widehat{\mathbf{A}} = \mathbf{A} - \mathbf{A}_{ex}$, where $\mathbf{A}_{ex}$ is related to the external magnetic field by $\nabla\times\mathbf{A}_{ex} = \mathbf{H}_{ex}$ and satisfies

$$\nabla\cdot\mathbf{A}_{ex} = 0\,, \qquad \mathbf{A}_{ex}\cdot\mathbf{n}|_{\partial\Omega} = 0\,.$$

By assuming $\widetilde{u}$ constant and $\mathbf{H}_{ex}$ independent of time and such that $\nabla\times\mathbf{H}_{ex} =$

0, the system (6.1)–(6.5) reduces to

$$\gamma(\psi_t - i\kappa\phi\psi) - \frac{1}{\kappa^2}\triangle\psi + \frac{2i}{\kappa}(\widehat{\mathbf{A}} + \mathbf{A}_{ex})\cdot\nabla\psi + |\widehat{\mathbf{A}} + \mathbf{A}_{ex}|^2\psi$$
$$+\psi\big(|\psi|^2 - 1 + \widehat{u} + \widetilde{u}\big) = 0\,, \tag{6.6}$$
$$\widehat{\mathbf{A}}_t - \nabla\phi + \nabla\times\nabla\times\widehat{\mathbf{A}} - \frac{i}{2\kappa}\big(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\big) + |\psi|^2(\widehat{\mathbf{A}} + \mathbf{A}_{ex}) = 0\,, \tag{6.7}$$
$$\alpha\widehat{u}_t - k\triangle\widehat{u} - \frac{1}{2}\big(\psi\bar{\psi}_t + \bar{\psi}\psi_t\big) = 0\,, \tag{6.8}$$
$$\nabla\psi\cdot\mathbf{n}|_{\partial\Omega} = 0\,, \qquad \widehat{\mathbf{A}}\cdot\mathbf{n}|_{\partial\Omega} = 0\,, \qquad (\nabla\times\widehat{\mathbf{A}})\times\mathbf{n}|_{\partial\Omega} = 0\,,$$
$$\nabla\phi\cdot\mathbf{n}|_{\partial\Omega} = 0\,, \qquad \widehat{u}|_{\partial\Omega} = 0\,, \tag{6.9}$$
$$\psi(x,0) = \psi_0(x)\,, \qquad \widehat{\mathbf{A}}(x,0) = \widehat{\mathbf{A}}_0(x)\,, \qquad \widehat{u}(x,0) = \widehat{u}_0(x)\,. \tag{6.10}$$

Let us denote by $L^p(\Omega)$, $p > 0$ and $H^s(\Omega)$, $s \in \mathbb{R}$, the usual Lebesgue and Sobolev spaces, endowed with the standard norms $\|\cdot\|_p$ and $\|\cdot\|_{H^s}$. In particular, we denote by $\|\cdot\|$ the norm in $L^2(\Omega)$. Given a time interval $[a,b]$ and a Banach space X, we denote by $C(a,b,X)$ $[L^p(a,b,X)]$ the space of continuous $[L^p]$ functions from $[a,b]$ into X, with the usual norms

$$\|f\|_{C(a,b,X)} = \sup_{t\in[a,b]}\|f(t)\|_X\,, \qquad \Big[\|f\|^p_{L^p(a,b,X)} = \int_a^b \|f(t)\|^p_X\Big]\,.$$

Finally let us introduce the following functional spaces

$$\mathcal{D}(\Omega) = \big\{\mathbf{A} : \mathbf{A}\in H^1(\Omega),\ \nabla\cdot\mathbf{A} = 0,\ \mathbf{A}\cdot\mathbf{n}|_{\partial\Omega} = 0\big\}\,,$$
$$H^1_m(\Omega) = \left\{\phi : \phi\in H^1(\Omega),\ \int_\Omega \phi\,dv = 0\right\}\,.$$

DEFINITION 6.1 *A triplet $(\psi, \widehat{\mathbf{A}}, \widehat{u})$ such that $\psi \in L^2(0,\tau,H^1(\Omega)) \cap H^1(0,\tau,\, L^2(\Omega))$, $\widehat{\mathbf{A}} \in L^2(0,\tau,\mathcal{D}(\Omega)) \cap H^1(0,\tau,H^1(\Omega)')$, $\widehat{u} \in L^2(0,\tau,H^1_0(\Omega)) \cap H^1(0,\tau,\, H^{-1}(\Omega))$, satisfying (6.10), is a weak solution of the problem (6.6)–(6.10) with $\phi \in L^2(0,\tau,H^1_m(\Omega))$, $\mathbf{A}_{ex} \in \mathcal{D}(\Omega)$ if*

$$\int_\Omega \Big[\gamma(\psi_t - i\kappa\phi\psi)\chi + \frac{1}{\kappa^2}\nabla\psi\cdot\nabla\chi - \frac{2i}{\kappa}\psi(\widehat{\mathbf{A}} + \mathbf{A}_{ex})\cdot\nabla\chi + |\widehat{\mathbf{A}} + \mathbf{A}_{ex}|^2\psi\chi$$
$$+\psi\chi(|\psi|^2 - 1 + \widehat{u} + \widetilde{u})\Big]dv = 0\,, \tag{6.11}$$
$$\int_\Omega \Big[\widehat{\mathbf{A}}_t\cdot\mathbf{b} + \phi\nabla\cdot\mathbf{b} + \nabla\times\widehat{\mathbf{A}}\cdot\nabla\times\mathbf{b} - \frac{i}{2\kappa}\big(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\big)\cdot\mathbf{b}$$
$$+|\psi|^2(\widehat{\mathbf{A}} + \mathbf{A}_{ex})\cdot\mathbf{b}\Big]dv = 0\,, \tag{6.12}$$
$$\int_\Omega \left[\alpha\widehat{u}_t v + k\nabla\widehat{u}\cdot\nabla v - \frac{1}{2}\big(\psi\bar{\psi}_t + \bar{\psi}\psi_t\big)v\right]dv = 0\,, \tag{6.13}$$

for each $\chi \in H^1(\Omega)$, $\mathbf{b} \in H^1(\Omega)$, $v \in H^1_0(\Omega)$ and for a.e. $t \in [0,\tau]$.

Notice that, since any $\mathbf{b} \in H^1(\Omega)$ can be decomposed as $\mathbf{b} = \mathbf{a} + \nabla\varphi$, with $\mathbf{a} \in \mathcal{D}(\Omega)$ and $\varphi \in H^2(\Omega)$, the equation (6.12) can be replaced by

$$\int_\Omega \Big[\widehat{\mathbf{A}}_t \cdot \mathbf{a} + \nabla \times \widehat{\mathbf{A}} \cdot \nabla \times \mathbf{a} - \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)\cdot \mathbf{a}$$
$$+|\psi|^2(\widehat{\mathbf{A}} + \mathbf{A}_{ex})\cdot \mathbf{a}\Big]dv = 0\,,$$
$$\int_\Omega \Big[\phi\triangle\varphi - \frac{i}{2\kappa}(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi)\cdot\nabla\varphi + |\psi|^2(\widehat{\mathbf{A}} + \mathbf{A}_{ex})\cdot\nabla\varphi\Big]dv = 0\,.$$

The following theorem proves the existence of the local solutions of the problem (6.6)–(6.10).

THEOREM 6.1 *Let $\psi_0 \in H^1(\Omega), \widehat{\mathbf{A}}_0 \in \mathcal{D}(\Omega), \widehat{u}_0 \in L^2(\Omega)$. Then there exist $\tau_0 > 0$ and a solution $(\psi, \widehat{\mathbf{A}}, \widehat{u})$ of the problem (6.6)–(6.10) in the time interval $(0, \tau_0)$. Moreover $\psi \in L^2(0, \tau_0, H^2(\Omega)) \cap C(0, \tau_0, H^1(\Omega))$, $\widehat{\mathbf{A}} \in L^2(0, \tau_0, H^2(\Omega)) \cap C(0, \tau_0, H^1(\Omega))$, $\widehat{u} \in C(0, \tau_0, L^2(\Omega))$.*

Proof. The proof is based on the Faedo-Galerkin method. Let $\chi_j, \mathbf{a}_j$ and v_j, $j \in \mathbb{N}$ be solutions of the boundary value problems

$$\begin{cases} -\triangle\chi_j = \lambda_j\chi_j \\ \nabla\chi_j \cdot \mathbf{n}|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} \nabla\times\nabla\times\mathbf{a}_j = \mu_j\mathbf{a}_j \\ \nabla\cdot\mathbf{a}_j = 0 \\ \mathbf{a}_j\cdot\mathbf{n}|_{\partial\Omega} = 0 \\ (\nabla\times\mathbf{a}_j)\times\mathbf{n}|_{\partial\Omega} = 0 \end{cases} \qquad \begin{cases} -\triangle v_j = \xi_j v_j \\ v_j|_{\partial\Omega} = 0 \end{cases}$$

where the eigenvalues λ_j, μ_j, ξ_j satisfy the inequalities $0 = \lambda_1 < \lambda_2 < ...$, $0 < \mu_1 < \mu_2 < ...$, $0 < \xi_1 < \xi_2 < ...$ and the eigenfunctions $\{\chi_j\}_{j\in\mathbb{N}}, \{\mathbf{a}_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ constitute orthonormal bases of $L^2(\Omega)$. Moreover $\chi_j \in H^1_m(\Omega)$ for each $j \geq 2$.

We denote by

$$\psi^m(x,t) = \sum_{j=1}^m \alpha_{jm}(t)\chi_j(x)\,, \qquad \widehat{\mathbf{A}}^m(x,t) = \sum_{j=1}^m \beta_{jm}(t)\mathbf{a}_j(x)\,,$$
$$\phi^m(x,t) = \sum_{j=1}^m \gamma_{jm}(t)\chi_j(x)\,, \qquad \widehat{u}^m(x,t) = \sum_{j=1}^m \delta_{jm}(t)v_j(x)\,,$$

which satisfy, for each $j = 1, ..., m$, the equations

$$\int_\Omega \Big[\gamma(\psi_t^m - i\kappa\phi^m\psi^m)\chi_j + \frac{1}{\kappa^2}\nabla\psi^m \cdot \nabla\chi_j - \frac{2i}{\kappa}\psi^m(\widehat{\mathbf{A}}^m + \mathbf{A}_{ex}) \cdot \nabla\chi_j$$
$$+|\widehat{\mathbf{A}}^m + \mathbf{A}_{ex}|^2\psi^m\chi_j + \psi^m\chi_j(|\psi^m|^2 - 1 + \widehat{u}^m + \widetilde{u})\Big] dv = 0\,, \quad (6.14)$$

$$\int_\Omega \Big[\widehat{\mathbf{A}}_t^m \cdot \mathbf{a}_j + \nabla \times \widehat{\mathbf{A}}^m \cdot \nabla \times \mathbf{a}_j - \frac{i}{2\kappa}\big(\psi^m\nabla\bar{\psi}^m - \bar{\psi}^m\nabla\psi^m\big) \cdot \mathbf{a}_j$$
$$+|\psi^m|^2(\widehat{\mathbf{A}}^m + \mathbf{A}_{ex}) \cdot \mathbf{a}_j\Big] dv = 0\,, \quad (6.15)$$

$$\int_\Omega \left[\alpha\widehat{u}_t^m v_j - \frac{1}{2}\big(\psi^m\bar{\psi}_t^m + \bar{\psi}^m\psi_t^m\big)v_j + k\nabla\widehat{u}^m \cdot \nabla v_j\right] dv = 0\,, \quad (6.16)$$

$$\int_\Omega \Big[\phi^m \triangle\chi_j - \frac{i}{2\kappa}\big(\bar{\psi}^m\nabla\psi^m - \psi^m\nabla\bar{\psi}^m\big) \cdot \nabla\chi_j$$
$$+|\psi^m|^2(\widehat{\mathbf{A}}^m + \mathbf{A}_{ex}) \cdot \nabla\chi_j\Big] dv = 0\,. \quad (6.17)$$

The function ϕ^m is supposed to verify the condition

$$\int_\Omega \phi^m dv = 0\,,$$

for all $t \in \mathbb{R}$. Moreover, since $\chi_j \in H_m^1(\Omega)$, $j \geq 2$, from the previous equation we deduce $\gamma_{1m} = 0$, for each $m \in \mathbb{N}$, so that

$$\phi^m(x,t) = \sum_{j=2}^{m} \gamma_{jm}(t)\chi_j(x)\,.$$

Let $(\psi_{0m}, \widehat{\mathbf{A}}_{0m}, \widehat{u}_{0m})$ be a sequence which converges to $(\psi_0, \widehat{\mathbf{A}}_0, \widehat{u}_0)$ with respect to the norm of $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ and denote by

$$\psi^m(x,0) = \psi_{0m}(x)\,, \qquad \widehat{\mathbf{A}}^m(x,0) = \mathbf{A}_{0m}(x)\,, \qquad \widehat{u}^m(x,0) = u_{0m}(x)\,.$$

Then the equations (6.14)–(6.16) constitute a system of ordinary differential equations for the unknowns α_{jm}, β_{jm} and δ_{jm} with initial conditions

$$\alpha_{jm}(0) = \int_\Omega \psi_{0m}\chi_j dv\,, \quad \beta_{jm}(0) = \int_\Omega \mathbf{A}_{0m} \cdot \mathbf{a}_j dv\,, \quad \delta_{jm}(0) = \int_\Omega u_{0m}v_j dv\,.$$

Notice that (6.17) allows to express γ_{jm}, $j \geq 2$, as a function of α_{jm}, β_{jm} and δ_{jm}.

Therefore the standard theory of ordinary differential equations ensures the existence and uniqueness of the local solutions.

By letting

$$\mathcal{F} = \gamma\|\psi^m\|_{H^1}^2 + \|\widehat{\mathbf{A}}^m\|^2 + \|\nabla \times \widehat{\mathbf{A}}^m\|^2 + \|\frac{i}{\kappa}\nabla\psi^m + \psi^m(\widehat{\mathbf{A}}^m + \mathbf{A}_{ex})\|^2$$
$$+\frac{1}{2}\||\psi^m|^2 - 1\|^2 + \alpha\|\widehat{u}^m\|^2 + 1,$$

the inequality

$$\frac{d\mathcal{F}}{dt}+\frac{\gamma}{2}\|\psi_t^m\|^2+\frac{1}{2\kappa^2}\|\triangle\psi^m\|^2+\frac{1}{2}\|\nabla\phi^m\|^2+\frac{k}{2}\|\nabla\widehat{u}^m\|^2+\|\widehat{\mathbf{A}}_t^m\|^2$$
$$+\|\nabla\times\widehat{\mathbf{A}}^m\|^2\le c\mathcal{F}^5 \tag{6.18}$$

can be proved. See [4] for details. An integration in $(0,t)$ leads to

$$\mathcal{F}\le(\mathcal{F}(0)^{-4}-ct)^{-1/4}\qquad t<\tau_0, \tag{6.19}$$

where τ_0 depends on the norms $\|\psi_{0m}\|_{H^1}, \|\mathbf{A}_{0m}\|_{H^1}, \|u_{0m}\|$. The previous inequalities allow to pass to the limit as $m\to\infty$ and prove the existence of a solution $(\psi,\mathbf{A},u)$ of the problem (6.6)–(6.10) satisfying $\psi\in C(0,\tau_0,H^1(\Omega))$, $\widehat{\mathbf{A}}\in C(0,\tau_0,H^1(\Omega))$ and $\widehat{u}\in C(0,\tau_0,L^2(\Omega))$.

The local solutions, defined in the time interval $(0,\tau_0)$ by Theorem 6.1, can be extended to the whole interval $(0,+\infty)$. Indeed we construct a Lyapunov functional for the system

$$\gamma f_t-\frac{1}{\kappa^2}\triangle f+(f^2-1+u)f-f|\mathbf{A}-\frac{1}{\kappa}\nabla\theta|^2=0\,, \tag{6.20}$$

$$\mathbf{A}_t-\nabla\phi+\nabla\times\nabla\times\mathbf{A}+f^2(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)=0\,, \tag{6.21}$$

$$\triangle\phi+\gamma f^2(\theta_t-\kappa\phi)=0\,, \tag{6.22}$$

$$\alpha u_t-ff_t-k\triangle u=0\,, \tag{6.23}$$

by multiplying the equations respectively by $f_t, \mathbf{A}_t-\kappa^{-1}\nabla\theta_t, -\phi+\kappa^{-1}\theta_t, \widehat{u}$ and integrating in Ω. We obtain

$$\|f_t\|^2+\frac{1}{2}\frac{d}{dt}\left[\frac{1}{\kappa^2}\|\nabla f\|^2+\frac{1}{2}\|f^2-1\|^2\right]+\int_\Omega ff_t\left[|\mathbf{A}-\frac{1}{\kappa}\nabla\theta|^2+u\right]dv=0,$$

$$\|\mathbf{A}_t\|^2+\frac{1}{2}\frac{d}{dt}\left(\|\nabla\times\mathbf{A}\|^2-2\int_{\partial\Omega}\mathbf{A}\times\mathbf{H}_{ex}\cdot\mathbf{n}da\right)$$
$$+\int_\Omega\left[f^2(\mathbf{A}-\frac{1}{\kappa}\nabla\theta)\cdot\left(\mathbf{A}_t-\frac{1}{\kappa}\nabla\theta_t\right)+\frac{1}{\kappa}\nabla\phi\cdot\nabla\theta_t\right]dv=0,$$

$$\|\nabla\phi\|^2+\gamma\kappa\|\frac{1}{\kappa}f\theta_t-f\phi\|^2-\frac{1}{\kappa}\int_\Omega\nabla\phi\cdot\nabla\theta_t dv=0,$$

$$\frac{\alpha}{2}\frac{d}{dt}\|\widehat{u}\|^2+k\|\nabla\widehat{u}\|^2-\int_\Omega\widehat{u}ff_t dv=0\,.$$

Adding the previous equations, we get

$$\frac{d\mathcal{G}}{dt}+\|f_t\|^2+\|\mathbf{A}_t\|^2+\|\nabla\phi\|^2+\gamma\kappa\|f(\frac{1}{\kappa}\theta_t-\phi)\|^2+k\|\nabla u\|^2=0\,, \tag{6.24}$$

where the functional $\mathcal{G}$ is defined as

$$\mathcal{G} = \frac{1}{2}\left(\frac{1}{\kappa^2}\|\nabla f\|^2 + \frac{1}{2}\|f^2 - 1\|^2 + \|f(\mathbf{A} - \frac{1}{\kappa}\nabla\theta)\|^2 + \|\nabla\times\mathbf{A}\|^2\right.$$

$$\left. -2\int_{\partial\Omega}\mathbf{A}\times\mathbf{H}_{ex}\cdot\mathbf{n}da + \nu\|\mathbf{H}_{ex}\|^2_{H^{-1/2}(\partial\Omega)} + \alpha\|u\|^2 + \int_\Omega \widetilde{u}f^2 dv\right)$$

and the constant ν is sufficiently large in order to make $\mathcal{G}$ positive.

The relation (6.24) yields

$$\mathcal{G}(t) \leq \mathcal{G}(0)\,, \qquad\qquad \forall t\geq 0,$$

which guarantees that the local solutions defined in $(0,\tau_0)$ can be extended in $(0,\infty)$. As a consequence of last inequality, we can prove some *a priori* estimates of the solutions. In particular, if the initial data are chosen such that the energy is finite, we have

$$\|f\|^2_{H^1} + \|\mathbf{A}\|^2_{H^1} + \|f\nabla\theta\|^2 + \|u\|^2 \leq C\,. \tag{6.25}$$

Moreover, by integrating the relation (6.24) in $(0,t)$ we obtain the further estimate

$$\int_0^t [\|f_t\|^2 + \|\mathbf{A}_t\|^2 + \|\nabla\phi\|^2 + \|f\theta_t\|^2 + \|\nabla u\|^2]ds \leq C\,. \tag{6.26}$$

The inequalities (6.25) and (6.26) lead to an estimate for the variable ψ

$$\|\psi\|^2_{H^1} + \int_0^t \|\psi_t\|^2 ds \leq C\,. \tag{6.27}$$

It can be proved ([3]) that if $f_0(x)\leq 1$ almost everywhere in Ω, then

$$f(x,t)\leq 1\,, \tag{6.28}$$

for all $t>0$. Accordingly, the relations (6.1), (6.25), (6.26) and (6.27) yield

$$\int_0^t \|\triangle\psi\|^2 ds \leq C\,. \tag{6.29}$$

THEOREM 6.2 *The solution $(\psi,\mathbf{A},u)$ of the system (6.1)–(6.5), with initial data $(\psi_0,\mathbf{A}_0,u_0)\in H^1(\Omega)\times\mathcal{D}(\Omega)\times L^2(\Omega)$ is unique.*

Proof. Let $(\psi_1,\mathbf{A}_1,u_1),(\psi_2,\mathbf{A}_2,u_2)$ be two solutions of the problem (6.6)–(6.10) with the same initial data $(\psi_0,\mathbf{A}_0,u_0)$ and sources $\mathbf{A}_{ex},\widetilde{u}$. By denoting by $\psi=\psi_1-\psi_2$, $\mathbf{A}=\mathbf{A}_1-\mathbf{A}_2$, $\phi=\phi_1-\phi_2$ and $u=u_1-u_2$, from the equations (6.6)–(6.8) and the inequalities (6.25)–(6.29) we deduce ([4])

$$\frac{1}{2}\frac{d}{dt}\left[\gamma\|\psi\|^2 + \frac{1}{\kappa^2}\|\nabla\psi\|^2 + \|\mathbf{A}\|^2 + \|\nabla\times\mathbf{A}\|^2 + \alpha\|u\|^2\right]$$

$$\leq \varphi_1(t)\|\psi\|^2_{H^1} + \varphi_2(t)\|\mathbf{A}\|^2_{H^1} + C\|u\|^2$$

where φ_1, φ_2 are L^1-functions of time. Therefore, an application of Gronwall's inequality proves $\psi = 0$, $\mathbf{A} = 0$, $u = 0$.

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Some global in time results for integrodifferential parabolic inverse problems

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Abstract We discuss a global in time existence and uniqueness result for an inverse problem arising in the theory of heat conduction for materials with memory. The novelty lies in the fact this is a global in time well posed problem in the sense of Hadamard, for semilinear parabolic inverse problems of integrodifferential type.

1 Introduction

In this paper we discuss some strategies we can use in the study of parabolic integrodifferential inverse problems. The choice of the strategy depends on what type of nonlinearities are involved. We consider the heat equation for materials with memory since it is one of the most important physical examples to which our methods apply. Other models, for instance in the theory of population dynamics, can also be considered within our framework. We recall, for the sake of completeness, the heat equation for materials with memory. Let Ω be an open and bounded set in $\mathbb{R}^3$ and T be a positive real number. The evolution equation for the temperature u is given, for $(t, x) \in [0, T] \times \Omega$, by

$$D_t u(t,x) = k\Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\, ds + F(u(t,x)), \tag{1.1}$$

where k is the diffusivity coefficient, h accounts for the memory effects and F is the heat source. In the inverse problem we consider, besides the temperature u, also h as a further unknown, and to determine it we add an additional measurement on u represented in integral form by

$$\int_\Omega \phi(x)u(t,x)\, dx = G(t), \qquad \forall t \in [0,T], \tag{1.2}$$

where ϕ and G are given functions representing the type of device used to measure u (on a suitable part of the body Ω) and the result of the measurement, respectively. We associate with (1.1)–(1.2) the initial-boundary conditions, for example of Neumann type:

$$\begin{cases} u(0,x) = u_0(x), & x \in \overline{\Omega}, \\ D_\nu u(t,x) = 0, & (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{1.3}$$

ν denoting the outward normal unit vector.
So one of the problems we are going to investigate is the following.

PROBLEM 1.1 (The Inverse Problem with two types of nonlinearities): *determine the temperature* $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$ *and the convolution kernel* $h : [0,T] \times \Omega \longrightarrow \mathbb{R}$ *satisfying* (1.1)–(1.3).

In the case when F is independent of u, but depends only on x and on t, we assume that the heat source is placed in a given position, but its time dependence is unknown, so we can suppose that

$$F(t,x) = f(t)g(x),$$

where f has to be determined and g is a given datum. Then we also assume that the diffusion coefficient k is unknown. The second inverse problem we will study is as follows.

PROBLEM 1.2 (An inverse problem with a nonlinearity of convolution type): *determine the temperature* $u : [0,T] \times \Omega \longrightarrow \mathbb{R}$*, the diffusion coefficient* k *and the functions* $h : [0,T] \longrightarrow \mathbb{R}$*,* $f : [0,T] \longrightarrow \mathbb{R}$ *satisfying the system*

$$\begin{cases} D_t u(t,x) = k\Delta u(t,x) + \int_0^t h(t-s)\Delta u(s,x)\,ds + f(t)g(x), \\ u(0,x) = u_0(x), \quad x \in \Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\Omega, \end{cases} \tag{1.4}$$

with the additional conditions

$$\int_\Omega u(t,x)\mu_j(dx) = G_j(t), \quad \forall t \in [0,T], \quad j = 1,2, \tag{1.5}$$

where g, u_0, G_1, G_2 are given data and μ_1 and μ_2 are finite Borel measures in $C(\overline{\Omega})$.

REMARK 1.1 The additional conditions considered for Problem 1.2 (cf. (1.5)) is more general than the one considered for Problem 1.1 (cf. (1.2)). This is due to the fact that in Problem 1.2 we will choose the space of continuous functions as reference space. Such a setting has the advantage to allow additional measurements of the temperature also on the boundary of Ω, while in the L^p-setting (if we consider additional measurements of type (1.5)), one is compelled to make further measurements inside the body. In fact, for Problem 1.2 the measure μ_j $(1 \leq j \leq 2)$ is Borel measure in $\overline{\Omega}$, e.g., concentrated on the surface $\partial\Omega$, while, in the other case, ϕ (cf. (1.2)) is an element of $L^{p'}(\Omega)$ with $1/p + 1/p' = 1$.

Several identification problems involving the heat equation with memory have been faced and solved in the recent years; see for example [4, 5, 6, 8, 9, 12, 14] and the literature therein. The type of results we find are theorems of *local in time* existence and uniqueness for the solution of the inverse problem considered. What is still an open problem is to find *global in time* existence and uniqueness theorems for a sufficiently large class of nonlinearities that involve the function $F(u(t,x))$.

Since in this paper we want to show what kind of difficulties we have to overcome to solve Problem 1.1 (Problem 1.2, even though it has more unknowns, from a technical view point is a particular case) we make the following classification of the difficulties one has to face when dealing with this kind of inverse problems.

The main difficulty arises because there are two types of nonlinearities: one is in the convolution term $\int_0^t h(t-s)\Delta u(s,x)\,ds$, while the second one is obviously due to the nonlinear function $F(u(t,x))$.

There are several papers in which nonlinearities of convolution type have been studied. In particular, in [10, 11] the authors prove global in time results, in suitable weighted spaces, for convolution kernels that depend also on one space variable.

Such spaces are the natural tool to face inverse problems in which there are only nonlinearities of convolution type.

The presence of the nonlinear function $F(u(t,x))$ of the unknown u leads us to look for *a priori* estimates for the unknowns u and h, so that from a local in time result we obtain a global in time one.

The problem that arises with both nonlinearities is that the weighted spaces are not suitable in treating the nonlinear term $F(u(t,x))$. What has been done in the recent paper [7] is to find methods that allow us to treat both nonlinearities simultaneously.

In the case where we are looking for a local in time solution there is a wide class of function spaces in which it is possible to set our problem; see for example [4, 5, 6, 12, 14, 20, 22], but in the case we have to find *a priori* estimates just a few spaces are useful to this aim.

In this paper we present global in time results in the space of bounded functions with values in an interpolation space for a problem that involves only the nonlinearity of convolution type and then we show the very recent results in the Sobolev setting in the case there are both type of nonlinearities.

In the literature we can find the recent paper [21], in which the authors prove a *conditioned* global in time result for a phase-field model using a different strategy with respect to ours that suits well for the particular coupling of the equations of the phase-field system they consider. This is due to the fact that the two types of nonlinearities belong to two different equations.

Our final target is to generalize the technique developed in [7] to the different phase-field models that we can find in the literature; see for example [2, 3, 16, 20, 24, 25].

Let us explain what are the main differences in dealing with one or two types of nonlinearities showing the strategies we use.

In the case when the term F is a given datum that does not depend on the temperature u, or as in Problem 1.2 where $F = fg$ with f unknown, we use the weighted spaces, to be introduced in the sequel, and we proceed as follows.

(1) In the case it is possible to formulate our problem in at least two function spaces, we consider an abstract formulation of the inverse problem relating it to a Banach space X. This is not strictly necessary if the results hold just in the case when X can be uniquely chosen.

(2) We choose a functional setting. For example we can take the space of bounded functions on $[0, T]$ or the Sobolev spaces on $[0, T]$ with values in the Banach space X and we select the related optimal regularity theorem for the linearized version of the problem.

(3) We prove that the abstract version of the problem is equivalent to a suitable fixed point system.

(4) Since the fixed point system contains integral operators, we have to estimate them in the weighted spaces we are considering (exponential weight $e^{\sigma t}$, $\sigma \in \mathbb{R}^+$, $t \in [0, T]$ is usually used). The estimates for the integral operators must be such that suitable constants depending on σ approaches zero as $\sigma \to \infty$.

(5) By the Contraction Principle we prove that the equivalent problem has a unique solution, so we get existence and uniqueness of a solution to our inverse problem.

(6) We apply the abstract results to the concrete problem.

Let us come to the doubly nonlinear case in which F depends on u. The main idea to solve the problem in this case is to prove that there exists a local in time solution of the inverse problem in Sobolev spaces without weights, then we linearize the convolution term and we find a priori estimates for u and for the convolution kernel h. More precisely we proceed as follows.

(a) In this case we do not give an abstract formulation since at the moment we are able to prove our results only in the Sobolev setting.

(b) We use the Sobolev spaces $W^{k,p}(0,T;L^p(\Omega))$.

(c) Analogue to (3), but the concrete system is considered instead of the abstract one.

(d) The fixed point system contains integral operators; we have to estimate them in the Sobolev spaces we have chosen.

(e) We apply the Contraction Principle to prove that there exists a unique *local in time* solution. Thanks to the equivalence theorem previously obtained we get *local in time* existence and uniqueness of the solution to our inverse problem. We prove a *global in time* uniqueness result without the condition that F_u be bounded.

(f) We linearize the convolution term thanks to the *local in time* existence and uniqueness theorem. We observe that a unique solution $(\hat{u}, \hat{h})$ exists in $[0,\tau]$ for some $\tau > 0$. We set $v_\tau(t) = v(\tau + t)$ and $h_\tau(t) = h(\tau + t)$ and consider, for $0 < t < \tau$, the splitting

$$\int_0^{\tau+t} h(\tau + t - s)\Delta v(s,x)ds = h_\tau * \Delta\hat{v}(t,x) + \hat{h} * \Delta v_\tau(t,x) + \tilde{F}(t,x),$$

where the symbol $*$ stands for the convolution (see below) and $\tilde{F}(t,x)$ is a given data depending on the known functions $(\hat{u}, \hat{h})$. This way of rewriting the convolution term allows us to avoid the weighted spaces that have a bad behavior when we deal with the nonlinearity $F(u)$.

(g) We deduce the a priori estimates for $v_\tau(t)$ and $h_\tau(t)$ for $0 < t < \tau$ with the condition F_u be bounded. In a finite number of steps we extend the solution to the interval $[0,T]$.

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2 Functional settings and preliminary material

Let X be a Banach space with norm $\|\cdot\|$ and let $T > 0$. We denote by $C([0,T];X)$ the usual space of continuous functions with values in X, while we denote by $\mathcal{B}([0,T];X)$ the space of bounded functions with values in X. $\mathcal{B}([0,T];X)$ will be endowed with the sup-norm

$$\|u\|_{\mathcal{B}([0,T];X)} := \sup_{0\le t\le T} \|u(t)\| \tag{2.1}$$

and $C([0,T];X)$ will be considered a closed subspace of $\mathcal{B}([0,T];X)$. We will use the notations $C([0,T];\mathbb{R}) = C([0,T])$ and $\mathcal{B}([0,T];\mathbb{R}) = \mathcal{B}([0,T])$. By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X into itself equipped with the sup–norm, while $\mathcal{L}(X;\mathbb{R}) = X'$ is the space of all bounded linear functionals on X considered with the natural norm. We set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. If $s \in \mathbb{Z}$, $s \geq 2$ we set $W_B^{s,p}(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv 0\}$. We denote by $B^s_{p,q}(\Omega)$ ($s > 0$, $1 \leq p, q \leq +\infty$) the Besov spaces. The symbol $(\cdot,\cdot)_{\theta,p}$ stands for the real interpolation functor ($0 < \theta < 1$, $1 \leq p \leq +\infty$). For all $h \in L^1(0,T)$ and $f : (0,T) \to X$ we define the convolution

$$h * f(t) := \int_0^t h(t-s)f(s)ds,$$

whenever the integral has a meaning. Let $p \in [1,+\infty)$, $m \in \mathbb{N}_0$; if $f \in W^{m,p}(0,T;X)$ (see [1]), we set

$$\|f\|_{W^{m,p}(0,T;X)} := \sum_{j=0}^{m-1} \|f^{(j)}(0)\|_X + \|f^{(m)}\|_{L^p(0,T;X)}.$$

For the sake of brevity we define the Banach space

$$X(T,p) = W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)),$$

where $T \in \mathbb{R}^+$, $p \in [1,+\infty]$. If $u \in X(T,p)$ we set

$$\|u\|_{X(T,p)} = \|u\|_{W^{1,p}(0,T;L^p(\Omega))} + \|u\|_{L^p(0,T;W^{2,p}(\Omega))}.$$

We now give the definition:

DEFINITION 2.1 *Let $A : \mathcal{D}(A) \subseteq X \to X$ be a linear operator, possibly with $\overline{\mathcal{D}(A)} \neq X$. Operator A is said to be sectorial if it satisfies the following assumptions:*

- *there exist $\theta \in (\pi/2, \pi)$ and $\omega \in \mathbb{R}$, such that any $\lambda \in \mathbb{C} \setminus \{\omega\}$ with $|\arg(\lambda - \omega)| \leq \theta$ belongs to the resolvent set of A.*
- *there exists $M > 0$ such that $\|(\lambda - \omega)(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M$ for any $\lambda \in \mathbb{C} \setminus \{\omega\}$ with $|\arg(\lambda - \omega)| \leq \theta$.*

The above definition of sectorial operator is important to define the semigroup of bounded linear operators $\{e^{tA}\}_{t\geq 0}$, in $\mathcal{L}(X)$, so that $t \to e^{tA}$ is an analytic function from $(0,+\infty)$ to $\mathcal{L}(X)$.
Let us define the family of interpolation spaces (see [23] or [29]) $\mathcal{D}_A(\theta,\infty)$, $\theta \in (0,1)$, between $\mathcal{D}(A)$ and X by

$$\mathcal{D}_A(\theta,\infty) = \Big\{x \in X : \ |x|_{\mathcal{D}_A(\theta,\infty)} := \sup_{0<t<1} t^{1-\theta}\|Ae^{tA}x\| < \infty\Big\} \tag{2.2}$$

with the norm

$$\|x\|_{\mathcal{D}_A(\theta,\infty)} = \|x\| + |x|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.3}$$

We also set

$$\mathcal{D}_A(1+\theta,\infty) = \{x \in \mathcal{D}(A) : Ax \in \mathcal{D}_A(\theta,\infty)\}. \tag{2.4}$$

$\mathcal{D}_A(1+\theta,\infty)$ turns out to be a Banach space when equipped with the norm

$$\|x\|_{\mathcal{D}_A(1+\theta,\infty)} = \|x\| + \|Ax\|_{\mathcal{D}_A(\theta,\infty)}. \tag{2.5}$$

For Problem 1.2 we will use the following optimal regularity result:

THEOREM 2.1 (Optimal regularity in spaces $\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$) *Let X be a Banach space. Consider the Cauchy Problem:*

$$(CP) \quad \begin{cases} u'(t) = Au(t) + f(t), & t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{2.6}$$

where $A : \mathcal{D}(A) \to X$ is a sectorial operator and $\theta \in (0,1)$. For any $f \in C([0,T];X) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))$, $u_0 \in \mathcal{D}_A(\theta+1,\infty)$ the Cauchy problem (CP) admits a unique solution $u \in C^1([0,T];X) \cap C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(\theta+1,\infty))$.

Proof. See the book [23] or the original paper [27].

As we have discussed in the introduction for Problem 1.1 at the moment we have the only possibility to choose Sobolev spaces, if we want a global in time result. For this reason we do not formulate the inverse problem in an abstract setting. As a consequence the optimal regularity result we are in need of is formulated just for the concrete case.

THEOREM 2.2 (Optimal regularity in spaces $X(T,p)$) *Let Δ be the Laplace operator and $k_0 \in \mathbb{R}^+$. Consider the problem*

$$\begin{cases} D_t u(t,x) = k_0 \Delta u(t,x) + F(t,x), & (t,x) \in [0,T] \times \Omega, \\ D_\nu u(t,x') = 0, & (t,x') \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega. \end{cases} \tag{2.7}$$

Then, if $p \in (1,+\infty)$, $F \in L^p(0,T;L^p(\Omega))$ and $u_0 \in (L^p(\Omega), W_B^{2,p}(\Omega))_{1-1/p,p}$, (2.7) has a unique solution $u \in X(T,p)$. Moreover, for all $T_0 \in \mathbb{R}^+$, there exists $C(T_0) \in \mathbb{R}^+$, such that, if $0 < T \le T_0$,

$$\|u\|_{X(T,p)} \le C(T_0)(\|F\|_{L^p(0,T;L^p(\Omega))} + \|u_0\|_{(L^p(\Omega),W_B^{2,p}(\Omega))_{1-1/p,p}}). \tag{2.8}$$

Proof. It is that of Theorem 8.1 in [15].

From Theorem 3.5 in [17], we have that, for $p \in (1, +\infty)$:

$$(L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p} = \begin{cases} B^{2(1-1/p)}_{p,p}(\Omega) & \text{if } 1 < p < 3, \\ \{f \in B^{2(1-1/p)}_{p,p}(\Omega) : D_\nu f \equiv 0\} & \text{if } 3 < p < +\infty. \end{cases} \tag{2.9}$$

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3 The main results

We present in the following subsections the results we have obtained in the case we deal only with the nonlinearity of convolution type and the case in which both nonlinearities are involved. The space of bounded functions is used in the first case, while the Sobolev setting is used in the second case.

3.1 The case of one nonlinearity of convolution type

We give the Inverse Problem 1.2 an abstract formulation and then we apply the abstract Theorem 3.1 to the concrete case.

PROBLEM 3.1 (Inverse Abstract Problem (IAP)) *Let A be a sectorial operator in X. Determine a positive number k and three functions u, h, f, such that*

$$(\alpha) \begin{cases} u \in C^2([0,T]; X) \cap C^1([0,T]; \mathcal{D}(A)), \\ D_t u \in \mathcal{B}([0,T]; \mathcal{D}_A(1+\theta, \infty)), \qquad D_t^2 u \in \mathcal{B}([0,T]; \mathcal{D}_A(\theta, \infty)), \end{cases}$$

(β) $h \in C([0,T])$,

(γ) $f \in C^1([0,T])$,

satisfying the system

$$\begin{cases} u'(t) = kAu(t) + \int_0^t h(t-s)Au(s)ds + f(t)g, \qquad t \in [0,T], \\ u(0) = u_0, \end{cases} \tag{3.1}$$

and the additional conditions:

$$\langle u(t), \phi_j \rangle = G_j(t), \quad t \in [0,T], \quad j = 1, 2, \tag{3.2}$$

where ϕ_j $(j = 1, 2)$ are given bounded linear functionals on X, and G_j, u_0, g are given data.

We study the (IAP) under the following assumptions:

(H1) $\theta \in (0,1)$, X is a Banach space and A is a sectorial operator in X.

(H2) $u_0 \in \mathcal{D}_A(1+\theta+\varepsilon, \infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H3) $g \in \mathcal{D}_A(\theta+\varepsilon, \infty)$, for some $\varepsilon \in (0, 1-\theta)$.

(H4) $\phi_j \in X'$, for $j = 1, 2$.

(H5) $G_j \in C^2([0, T])$, for $j = 1, 2$.

(H6) The matrix

$$M := \begin{pmatrix} < Au_0, \phi_1 > & < g, \phi_1 > \\ < Au_0, \phi_2 > & < g, \phi_2 > \end{pmatrix} \tag{3.3}$$

is invertible and its inverse is defined by

$$M^{-1} := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \tag{3.4}$$

(H7) We require that the linear system

$$\begin{cases} k_0 < Au_0, \phi_1 > + f_0 < g, \phi_1 >= G_1'(0), \\ k_0 < Au_0, \phi_2 > + f_0 < g, \phi_2 >= G_2'(0), \end{cases} \tag{3.5}$$

has a unique solution (k_0, f_0) with $k_0 > 0$.

(H8) $v_0 := k_0 Au_0 + f_0 g \in \mathcal{D}_A(1+\theta, \infty)$.

(H9) $< u_0, \phi_j >= G_j(0), \qquad < v_0, \phi_j >= G_j'(0), \qquad j = 1, 2.$

REMARK 3.1 Owing to (H6) the first component k_0 of the solution (k_0, f_0) is positive if and only if

$$\frac{1}{\det M}[G_1'(0) < g, \phi_2 > - G_2'(0) < g, \phi_1 >] > 0.$$

The main abstract result is the following:

THEOREM 3.1 *Assume that conditions (H1)–(H9) are fulfilled. Then Problem 3.1 has a unique (global in time) solution (k, u, h, f), with $k \in \mathbb{R}^+$, and u, h, f satisfying conditions (α), (β) and (γ).*

Proof. See Section 4 for the main steps of the proof or Section 5 in [13] for all the details.

An application to the concrete case. We choose as reference space

$$X = C(\overline{\Omega}), \tag{3.6}$$

where Ω is an open bounded set in $\mathbb{R}^n$ (in the introduction we have considered the physical case $n = 3$, but the result holds for any $n \in \mathbb{N}$) with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$. We define

$$\begin{cases} \mathcal{D}(A) = \Big\{ u \in \bigcap_{1 \le p < +\infty} W^{2,p}(\Omega) \ : \Delta u \in C(\overline{\Omega}), \quad D_\nu u|_{\partial\Omega} = 0 \Big\}, \\ Au := \Delta u, \quad \forall u \in \mathcal{D}(A). \end{cases} \tag{3.7}$$

It was proved by Stewart (see [28]) that A is a sectorial operator in X. Then we recall the following characterizations concerning the interpolation spaces related to A (see [23]):

$$\mathcal{D}_A(\xi, \infty) = \begin{cases} C^{2\xi}(\overline{\Omega}), \quad \text{if} \ \ 0 < \xi < 1/2, \\ \{u \in C^{2\xi}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = 0\}, \quad \text{if} \ \ 1/2 < \xi < 1. \end{cases} \tag{3.8}$$

Consequently, if $0 < \xi \le \theta + \varepsilon$, we have

$$\mathcal{D}_A(1 + \xi, \infty) = \{u \in C^{2(1+\xi)}(\overline{\Omega}) : D_\nu u|_{\partial\Omega} = 0\}. \tag{3.9}$$

So we consider the Inverse Problem 1.2 under the following assumptions:

(K1) Ω is an open bounded set in $\mathbb{R}^n$ with boundary of class $C^{2(1+\theta+\varepsilon)}$, for some $\theta \in (0, 1/2)$, $\varepsilon \in (0, (1/2) - \theta)$.

(K2) $u_0 \in C^{2(1+\theta+\varepsilon)}(\overline{\Omega})$, $\quad D_\nu u_0|_{\partial\Omega} = 0$.

(K3) $g \in C^{2(\theta+\varepsilon)}(\overline{\Omega})$.

(K4) For $j = 1, 2$, μ_j is a bounded Borel measure in $\overline{\Omega}$. We set, for $\psi \in X$,

$$< \psi, \phi_j >:= \int_{\overline{\Omega}} \psi(x) \mu_j(dx). \tag{3.10}$$

(K5) Suppose that (H5) holds.

(K6) Suppose that (H6) holds with ϕ_j $(j = 1, 2)$ defined in (3.10) and A defined in (3.7).

(K7) Suppose that (H7) holds.

(K8) $v_0 := k_0 \Delta u_0 + f_0 g \in C^{2(1+\theta)}(\overline{\Omega})$, $\quad D_\nu v_0|_{\partial\Omega} = 0$.

(K9) Suppose that (H9) holds.

Applying Theorem 3.1 we immediately deduce:

THEOREM 3.2 *Assume that conditions (K1)–(K9) are satisfied. Then the Inverse Problem 1.2 has a unique (global in time) solution* (k, u, h, f)*, such* *that*

$$u \in C^2([0,T];C(\overline{\Omega})) \cap C^1([0,T];\mathcal{D}(A)),$$
$$D_t u \in \mathcal{B}([0,T];C^{2(1+\theta)}(\overline{\Omega})), \quad D_t^2 u \in \mathcal{B}([0,T];C^{2\theta}(\overline{\Omega})),$$
$$h \in C([0,T]), \qquad f \in C^1([0,T]), \qquad k \in \mathbb{R}^+,$$

A being defined in (3.7).

3.2 The case of two nonlinearities

We solve the Inverse Problem 1.1 under the following conditions on the data:

(I1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2.

(I2) $p \in (1, +\infty)$, $n \in \mathbb{N}$, with $n < 2p$.

(I3) $u_0 \in W_B^{2,p}(\Omega)$.

(I4) $\phi \in W^{2,p'}(\Omega)$. We set $\psi := \Delta\phi$.

(I5) $F \in C^\infty(\mathbb{R})$.

(I6) $v_0 := \Delta u_0 + F(u_0) \in (L^p(\Omega), W_B^{2,p}(\Omega))_{1-1/p,p}$.

(I7) $g \in W^{2,p}(0,T)$.

(I8) $\Phi(u_0) = g(0)$ and $\Phi(v_0) = g'(0)$.

(I9) $\Phi(\Delta u_0) := \int_\Omega \phi(x)\Delta u_0(x)dx \neq 0$.

(I10) F_u is bounded.

THEOREM 3.3 (Global in time). *Let the assumptions (I1)–(I10) hold. Let $T > 0$ and $p \geq 2$. Then Problem 1.1 has a unique solution*

$$(u,h) \in [W^{2,p}(0,T;L^p(\Omega)) \cap W^{1,p}(0,T;W^{2,p}(\Omega))] \times L^p(0,T).$$

Proof. See Section 5 for the main steps of the proof or Section 7 in [7] for all the details.

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4 The strategy for nonlinearity of convolution type in weighted spaces

We now want to show in a more explicit way how we obtain our results sketching some proofs. We follow the list in the introduction.

Step (1). The abstract formulation is Problem 3.1.

Step (2). The functional setting is the space of bounded function with values in an interpolation space, see (α) for the function u.

Step (3). We reformulate Problem 3.1 in terms of the equivalent nonlinear fixed point system (4.10). In proving Theorem 4.1 we find out a set of regularity and compatibility conditions on the data that makes the inverse problem well posed. To this aim, we start by introducing some notations. We set

$$A_0 := k_0 A. \tag{4.1}$$

As $k_0 > 0$, see (H7), A_0 is a sectorial operator in X. Next, we set, for $t \in [0,T]$,

$$\overline{h}_1(t) := a_{11} G_1''(t) + a_{12} G_2''(t), \tag{4.2}$$

$$\overline{w}_1(t) := a_{21} G_1''(t) + a_{22} G_2''(t), \tag{4.3}$$

$$\overline{v}(t) := e^{tA_0} v_0. \tag{4.4}$$

We immediately observe that $\overline{h}_1$ and $\overline{w}_1$ belong to $C([0,T])$ and, owing to Theorem 2.1, $\overline{v} \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))$. Next, we define, again for $t \in [0,T]$,

$$\overline{h}(t) := \overline{h}_1(t) - k_0 a_{11} < A\overline{v}(t), \phi_1 > - k_0 a_{12} < A\overline{v}(t), \phi_2 >, \tag{4.5}$$

$$\overline{w}(t) := \overline{w}_1(t) - k_0 a_{21} < A\overline{v}(t), \phi_1 > - k_0 a_{22} < A\overline{v}(t), \phi_2 > . \tag{4.6}$$

Of course, $\overline{h}$ and $\overline{w}$ belong to $C([0,T])$. Finally, we introduce the following (nonlinear) operators, defined for every $(v,h,w) \in [C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty))] \times C([0,T]) \times C([0,T])$:

$$\mathcal{R}_1(v,h,w)(t) := e^{tA_0} * (hAu_0 + wg + h * Av)(t), \tag{4.7}$$

$$\begin{aligned}
\mathcal{R}_2(v,h,w)(t) := & -k_0[a_{11} < A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\
& + a_{12} < A\mathcal{R}_1(v,h,w)(t), \phi_2 >] \\
& - a_{11} < h * A\overline{v}(t), \phi_1 > \\
& - a_{12} < h * A\overline{v}(t), \phi_2 > \\
& - a_{11} < h * A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\
& - a_{12} < h * A\mathcal{R}_1(v,h,w)(t), \phi_2 >,
\end{aligned} \tag{4.8}$$

$$\begin{aligned}\mathcal{R}_3(v,h,w)(t) := -k_0[&a_{21} < A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\ &+a_{22} < A\mathcal{R}_1(v,h,w)(t), \phi_2 >] \\ &-a_{21} < h * A\overline{v}(t), \phi_1 > \\ &-a_{22} < h * A\overline{v}(t), \phi_2 > \\ &-a_{21} < h * A\mathcal{R}_1(v,h,w)(t), \phi_1 > \\ &-a_{22} < h * A\mathcal{R}_1(v,h,w)(t), \phi_2 > . \end{aligned} \tag{4.9}$$

We observe that $\mathcal{R}_2$ and $\mathcal{R}_3$ are well defined, because

$$\mathcal{R}_1(v,h,w) \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}_A(1+\theta,\infty)).$$

Moreover, $\mathcal{R}_2(v,h,w)$ and $\mathcal{R}_3(v,h,w)$ both belong to $C([0,T])$.
Now we can introduce the following problem:

PROBLEM 4.1 *Determine three functions v, h, w, such that*

(α') $v \in C([0,T];\mathcal{D}(A)) \cap \mathcal{B}([0,T];\mathcal{D}(1+\theta,\infty))$,

(β') $h \in C([0,T])$,

(γ') $w \in C([0,T])$,

satisfying the system

$$\begin{cases} v &= \overline{v} + \mathcal{R}_1(v,h,w), \\ h &= \overline{h} + \mathcal{R}_2(v,h,w), \\ w &= \overline{v} + \mathcal{R}_3(v,h,w). \end{cases} \tag{4.10}$$

THEOREM 4.1 (Equivalence) *Let A be a sectorial operator in the Banach space X and $\theta \in (0,1)$. Let us assume that the data g, u_0, ϕ_j and G_j $(j = 1,2)$ satisfy the conditions (H1)–(H9). Suppose that k, u, f, h satisfy Problem 3.1, with $k \in \mathbb{R}_+$ and u, f, h fulfilling the regularity conditions (α), (β), (γ). Then*

(I) $k = k_0$, the triplet (v,h,w), where $v = u'$, $w = f'$ satisfies the conditions (α'), (β'), (γ') and solves Problem 4.1;

(II) conversely, if (v,h,w), with the above regularity, is a solution of the Problem 4.1, then the triplet (u,h,f), where $u = u_0+1*v$, $f = f_0+1*w$, satisfies the regularity conditions (α), (β), (γ) and solves Problem 3.1 with $k = k_0$.

Proof. It is Theorem 4.2 in [13] and it is based on Theorem 2.1.

Step (4). *Fundamental estimates for the integral operators $\mathcal{R}_j$, $j = 1, 2, 3$, in the weighted space.*

We introduce some crucial estimates that will be essential to obtain global in time existence and uniqueness of a solution to our inverse problem. Let $\lambda > 0$, $T > 0$, $\theta \in (0, 1)$. If $f \in \mathcal{B}([0, T]; X)$, we set

$$\|u\|_{\mathcal{B}_\lambda([0,T];X)} := \sup_{0 \le t \le T} e^{-\lambda t} \|u(t)\|. \tag{4.11}$$

We denote by $C_\lambda([0, T]; X)$ the space $C([0, T]; X)$ equipped with the norm $\| \cdot \|_{\mathcal{B}_\lambda([0,T];X)}$.
We will use the notations $C_\lambda([0, T]; \mathbb{R}) = C_\lambda([0, T])$ and $\mathcal{B}_\lambda([0, T]; \mathbb{R}) = \mathcal{B}_\lambda([0, T])$. We now state some useful estimates in these weighted spaces for the solution of the Cauchy problem given by Theorem 2.1. We will list in the following theorems what kind of estimates we are in need of.

THEOREM 4.2 *Let $A : \mathcal{D}(A) \to X$ be a sectorial operator, $\theta \in (0, 1)$. Let us suppose that $f \in C([0, T]; X) \cap \mathcal{B}([0, T]; \mathcal{D}_A(\theta, \infty))$. Then the following estimates hold:*

$$\|e^{tA} * f\|_{C_\lambda([0,T];X)} \le \frac{C_0}{1 + \lambda} \|f\|_{C_\lambda([0,T];X)}; \tag{4.12}$$

if $\theta \le \xi \le 1 + \theta$,

$$\|e^{tA} * f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\xi,\infty))} \le \frac{C(\theta, \xi)}{(1 + \lambda)^{1+\theta-\xi}} \|f\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \tag{4.13}$$

with C_0 and $C(\theta, \xi)$ independent of f and λ.

Proof. It is that of Theorem 4.3 in [13].

THEOREM 4.3 *Let $\lambda \ge 0$, $h \in C([0, T])$ and $u \in C([0, T]; X) \cap \mathcal{B}([0, T]; \mathcal{D}_A(\theta, \infty))$. Then there exists $C > 0$, independent of T, λ, h, u, such that, if*

$$h * u(t) := \int_0^t h(t - s) u(s) ds, \tag{4.14}$$

$h * u \in C([0, T]; X) \cap \mathcal{B}([0, T]; \mathcal{D}_A(\theta, \infty))$ and satisfies the following estimate:

$$\begin{aligned} \|h * u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \le C \min \Big\{ & T \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))}, \\ & (1 + \lambda)^{-1} \|h\|_{\mathcal{B}_\lambda([0,T])} \|u\|_{\mathcal{B}([0,T];\mathcal{D}_A(\theta,\infty))}, \\ & (1 + \lambda)^{-1} \|h\|_{\mathcal{B}([0,T])} \|u\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \Big\}. \end{aligned} \tag{4.15}$$

Proof. It is that of Theorem 4.4 in [13].

The above theorems give us the possibility to estimate the operators $\mathcal{R}_j$, $j = 1, 2, 3$.

LEMMA 4.1 *Assume that conditions (H1)–(H9) are satisfied. Let $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ be the operators defined in (4.7), (4.7), (4.8), respectively. Then there exists $C > 0$ such that, for all $\lambda \geq 0$, $v, v_1, v_2 \in C([0,T]; \mathcal{D}(A)) \cap \mathcal{B}([0,T]; \mathcal{D}_A(1+\theta,\infty))$, $h, h_1, h_2 \in C([0,T])$ and $w, w_1, w_2 \in C([0,T])$ we have*

$$\begin{aligned}\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \leq C[&\|\overline{h} * A\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(\theta,\infty))} \\ &+(1+\lambda)^{-\varepsilon}(\|h\|_{\mathcal{B}_\lambda([0,T])} + \|w\|_{\mathcal{B}_\lambda([0,T])}) \\ &+(1+\lambda)^{-1}(\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])} \\ &+\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}) \\ &+\|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.16}$$

$$\begin{aligned}&\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad \leq C[\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|v_1-\overline{v}\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + \|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + (1+\lambda)^{-\varepsilon}(\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} + \|w_1-w_2\|_{\mathcal{B}_\lambda([0,T])}) \\ &\qquad + (1+\lambda)^{-1}\|v_1-v_2\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.17}$$

and, if $i \in \{2,3\}$,

$$\begin{aligned}&\|\mathcal{R}_i(v,h,w)\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad \leq C[\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} + (1+\lambda)^{-1}\|h\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad + \|h-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v,h,w)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}];\end{aligned} \tag{4.18}$$

$$\begin{aligned}&\|\mathcal{R}_i(v_1,h_1,w_1) - \mathcal{R}_i(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad \leq C[\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &\qquad + (1+\lambda)^{-1}\|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])} \\ &\qquad + \|h_1-h_2\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))} \\ &+\|h_2-\overline{h}\|_{\mathcal{B}_\lambda([0,T])}\|\mathcal{R}_1(v_1,h_1,w_1) - \mathcal{R}_1(v_2,h_2,w_2)\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}].\end{aligned} \tag{4.19}$$

Proof. It is that of Lemma 5.1 in [13] and it is based on Theorems 4.2 and 4.3.

Step 5. By the Contraction Principle we prove Theorem 3.1. Let $\lambda \geq 0$. We set

$$Y(\lambda) := (C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty)) \times C_\lambda([0,T])^2, \quad (4.20)$$

and we endow it with the norm

$$\|(v,h,w)\|_{Y(\lambda)} := \max\{\|v\|_{\mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))}, \|h\|_{\mathcal{B}_\lambda([0,T])}, \|w\|_{\mathcal{B}_\lambda([0,T])}\}, \quad (4.21)$$

with $v \in C_\lambda([0,T];\mathcal{D}(A)) \cap \mathcal{B}_\lambda([0,T];\mathcal{D}_A(1+\theta,\infty))$, $h \in C_\lambda([0,T])$ and $w \in C_\lambda([0,T])$. With the norm (4.21) $Y(\lambda)$ becomes a Banach space.
Let $\lambda \geq 0$, $\rho > 0$ and set

$$Y(\lambda,\rho) := \{(v,h,w) \in Y(\lambda) : \|(v,h,w) - (\overline{v},\overline{h},\overline{w})\|_{Y(\lambda)} \leq \rho\}. \quad (4.22)$$

Then, for every $\rho > 0$, $Y(\lambda,\rho)$ is a closed subset of $Y(\lambda)$.
Now we introduce the following operator N: if $(v,h,w) \in Y(\lambda)$, we set

$$N(v,h,w) := (\overline{v} + \mathcal{R}_1(v,h,w), \overline{h} + \mathcal{R}_2(v,h,w), \overline{w} + \mathcal{R}_3(v,h,w)). \quad (4.23)$$

Clearly N is a nonlinear operator in $Y(\lambda)$.
Now we show that Problem 3.1 has a solution (k,u,h,f), with the regularity properties (α), (β), (γ).
Applying Theorem 4.1, we are reduced to looking for a solution (v,h,w) of system (4.10), satisfying the conditions (α'), (β'), (γ'). This is equivalent to looking for a fixed point of N in $Y(\lambda)$, for some $\lambda \geq 0$. For the details see Section 5 in [13].

Step (6). An application of the abstract result is Theorem 3.2.

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5 The strategy for the case of two nonlinearities in Sobolev spaces

We show the main ideas on which is based the global in time result for the doubly nonlinear problem.

Step (a)–(b). We consider in this case the concrete formulation of the problem since the correct functional setting is the Sobolev space

$$X(T,p) = W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)).$$

Step (c). An equivalent reformulation of the problem is the following:

THEOREM 5.1 *Let the assumptions (I1)–(I9) hold. Let u and h verify the conditions*

$$u \in W^{2,p}([0,T];L^p(\Omega)) \cap W^{1,p}([0,T];W^{2,p}(\Omega)), \quad h \in L^p([0,T]), \quad (5.1)$$

and solve the system (1.1)–(1.3). We set $v := D_t u$, so v and h satisfy the conditions

$$v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega)), \quad h \in L^p([0,T]). \tag{5.2}$$

Then v and h solve the system (5.4) and (5.6). On the other hand, let v, h satisfy the conditions (5.2) and solve the system (5.4) and (5.6). If we set $u := u_0 + 1 * v$, then u, h verify the conditions (5.1) and solve the system (1.1)–(1.3).

Proof. We split the proof into two steps.

Step 1. Suppose that the problem (1.1)–(1.3) has a solution

$$u \in W^{2,p}(0,T;L^p(\Omega)) \cap W^{1,p}(0,T;W^{2,p}(\Omega)), \quad h \in L^p(0,T).$$

We set

$$D_t u(t,x) := v(t,x), \tag{5.3}$$

and we differentiate the first equation in (1.1) to get

$$\begin{cases} D_t v(t,x) = \Delta v(t,x) + h(t)\Delta u_0 + h * \Delta v(t,x) \\ \qquad\qquad + F_u(u_0(x) + 1 * v(t,x))v(t,x), \\ v(0,x) := v_0 = Au_0(x) + F(u_0(x)),\ x \in \Omega, \\ D_\nu v(t,x) = 0, \quad t \in [0,T],\ x \in \partial\Omega. \end{cases} \tag{5.4}$$

If (I1)–(I9) hold, obviously we have $v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ and $h \in L^p(0,T)$. Apply now functional Φ (cf. (I9)) to the first equation and, keeping in mind that $\Phi(D_t u)(t) = g'(t)$ and $\Phi(D_t^2 u)(t) = g''(t)$, we get

$$\begin{aligned} g''(t) = {} & \Phi[\Delta v(t,\cdot)] + h(t)\Phi[\Delta u_0(\cdot)] \\ & + h * \Phi[\Delta v(t,\cdot)] + \Phi[F_u(u_0(\cdot) + 1 * v(t,\cdot))v(t,\cdot)]. \end{aligned} \tag{5.5}$$

We can write, setting $\chi^{-1} := \Phi[\Delta u_0(\cdot)] \neq 0$,

$$\begin{aligned} h(t) = {} & \chi g''(t) - \chi\Phi[F_u(u_0(\cdot) + 1 * v(t,\cdot))v(t,\cdot)] \\ & - \chi\Phi[\Delta v(t,\cdot)] - \chi h * \Phi[\Delta v(t,\cdot)]. \end{aligned} \tag{5.6}$$

Step 2. Suppose now that $v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))$ and $h \in L^p(0,T)$ satisfy system (5.4) and (5.6). Since $D_t u(t,x) := v(t,x)$ we observe that the first equation in (5.4) can be rewritten as

$$D_t[D_t u(t,x) - \Delta u(t,x) - h * \Delta u(t,x) - F(u(t,x))] = 0,$$

which gives

$$D_t u(t,x) - \Delta u(t,x) - h * \Delta u(t,x) - F(u(t,x)) = C(x). \tag{5.7}$$

Setting $t = 0$, we have

$$v_0(x) - \Delta u_0(x) - f(u_0(x)) = C(x),$$

so we get $C = 0$. So, from equation (5.7) we deduce the first equation in (1.1). Consider the equation for h in (5.5), it can be written as:

$$D_t g'(t) = D_t[\Phi[\Delta u(t,\cdot)] + h * \Phi[\Delta u(t,\cdot)] + \Phi[F(u(t,\cdot))]]. \tag{5.8}$$

This gives

$$c + g'(t) = \Phi[\Delta u(t,\cdot)] + h * \Phi[\Delta u(t,\cdot)] + \Phi[F(u(t,\cdot))].$$

At $t = 0$ we have

$$c + g'(0) = \Phi[\Delta u_0(\cdot) + F(u_0(\cdot))].$$

Since $\Delta u_0(x) + F(u_0(x)) = v_0$ and $g'(0) = \Phi[v_0]$ we get $c = 0$. Then the equation

$$D_t \Phi[u(t,\cdot)] = g'(t)$$

becomes

$$c' + \Phi[u(t,\cdot)] = g(t).$$

Setting $t = 0$ and recalling the compatibility condition $\Phi[u(0)] = g(0)$, we get $c' = 0$ so that

$$\Phi[u(t,\cdot)] = g(t).$$

Step (d). We get the preliminary lemmas that are necessary to estimate the operators entering in the equivalent reformulation of the problem.

THEOREM 5.2 *Let X be a Banach space, $p \in (1,+\infty)$, $\tau \in \mathbb{R}^+$, $h \in L^p(0,\tau)$, $f \in L^p(0,\tau;X)$. Then $h * f \in L^p(0,\tau;X)$ and*

$$\|h * f\|_{L^p(0,\tau;X)} \leq \tau^{1-1/p} \|h\|_{L^p(0,\tau)} \|f\|_{L^p(0,\tau;X)}.$$

Proof. It is that of Theorem 3.2 in [7].

THEOREM 5.3 *Let X be a Banach space, $\tau \in \mathbb{R}^+$, $p \in (1,+\infty)$, $z \in W^{1,p}(0,\tau;X)$, with $z(0) = 0$. Then*

$$\|z\|_{L^\infty(0,\tau;X)} \leq \tau^{1-1/p} \|z\|_{W^{1,p}(0,\tau;X)}, \tag{5.9}$$

$$\|z\|_{L^p(0,\tau;X)} \leq p^{-1/p} \tau \|z\|_{W^{1,p}(0,\tau;X)}. \tag{5.10}$$

Proof. It is that of Theorem 3.3 in [7].

THEOREM 5.4 *Under the conditions* $(I1)$ *and* $(I2)$, $W^{2,p}(\Omega)$ *is continuously embedded in* $C(\overline{\Omega})$ *and is a space of pointwise multipliers for* $W^{s,p}(\Omega)$, *for* $s = 0, 1, 2$.

Proof. It is that of Theorem 3.4 in [7].

THEOREM 5.5 *Under the assumptions* $(I1)$,$(I2)$ *and* $(I3)$, *if* $S \in C^\infty(\mathbb{R})$, *then the map* $v \to S \circ v$ *is of class* C^∞ *from* $W^{2,p}(\Omega)$ *into itself. Moreover, for all* $k \in \mathbb{N}_0$, $(S \circ \cdot)^{(k)}$ *is bounded with values in* $\mathcal{L}^k(W^{2,p}(\Omega), W^{2,p}(\Omega))$ *in every bounded subset of* $W^{2,p}(\Omega)$.

Proof. It is that of Theorem 3.5 in [7].

THEOREM 5.6 *Assume that* $(I1)$ *and* $(I2)$ *are satisfied,* $S \in C^\infty(\mathbb{R})$, $u_0 \in W^{2,p}(\Omega)$. *Let* $R \in \mathbb{R}^+$, $0 < \tau \leq T$ *and let* V_1 *and* V_2 *be elements of* $X(\tau, p)$ *such that*

$$\max_{j \in \{1,2\}} \|V_j\|_{L^p(0,\tau;W^{2,p}(\Omega))} \leq R.$$

Then

$$\|S(u_0 + 1 * V_1)V_1 - S(\chi_0 + 1 * V_2)V_2\|_{L^p(0,\tau;L^p(\Omega))}$$
$$\leq C(R,T)\tau^{(p-1)/(2p)}\|V_1 - V_2\|_{X(\tau,p)}.$$

Proof. It is that of Theorem 3.6 in [7].

THEOREM 5.7 *Let* $p \in (1, +\infty)$, Ω *satisfying* $(H1)$, $\phi \in L^{p'}(\Omega)$, $\tau \in \mathbb{R}^+$. *We define in* $L^p(0,\tau; L^p(\Omega))$ *the operator*

$$\Phi[f](t) := \int_\Omega \phi(x) f(t,x) dx.$$

If $u \in X(\tau, p)$, *we consider the map* $u \to \Phi[\Delta u]$. *Then* $\Phi[\Delta u] \in L^p(0,\tau)$ *and*

$$\|\Phi[\Delta u]\|_{L^p(0,\tau)} \leq \omega(\tau)\|u\|_{X(\tau,p)}, \tag{5.11}$$

with $\omega(\tau) > 0$, *independent of* u, *and* $\lim_{\tau \to 0} \omega(\tau) = 0$.

Proof. It is that of Theorem 3.7 in [7].

Step (e). *The local in time existence theorem and the global in time uniqueness theorem without the condition* F_u *bounded.*

THEOREM 5.8 (Local in time existence and uniqueness). *Let the assumptions (I1)–(I9) hold. Then there exists $\tau \in (0,T]$, depending on the data, such that the inverse problem (1.1)–(1.3) has a unique solution*

$$(u,h) \in [W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega))] \times L^p(0,\tau).$$

Proof. It is that of Theorem 2.1 in [7] and its proof is based on Theorems 2.2 and 5.2–5.7.

THEOREM 5.9 (Global in time uniqueness). *Let the assumptions (I1)–(I9) hold. If $\tau \in (0,T]$, and if the inverse problem in Definition 1.2 has two solutions $(u_j,h_j) \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega)) \times L^p(0,\tau)$, $j \in \{1,2\}$, then $u_1 = u_2$ and $h_1 = h_2$.*

Proof. It is that of Theorem 2.2 in [7] and its proof is based on Theorems 2.2 and 5.2–5.7.

Step (f). *We linearize the convolution term.* From the local in time existence and uniqueness theorem we observe that a unique solution $\widehat{u}$, $\widehat{h}$ exists in $[0,\tau]$ for some $\tau > 0$. So we can consider the equations:

$$\begin{cases} D_t v(\tau+t,x) = \Delta v(\tau+t,x) + h(\tau+t)\Delta u_0 \\ + \int_0^{\tau+t} h(\tau+t-s)\Delta v(s,x)ds + F_u(u_0(x) + 1 * v(\tau+t,x))v(\tau+t,x), \\ v(\tau,x) = u_\tau(x), \quad x \in \Omega, \\ D_\nu v(\tau+t,x) = 0, \quad t \in [0,T],\ x \in \partial\Omega, \\ \Phi[v(\tau+t,\cdot)] := \int_\Omega \phi(x)v(\tau+t,x)dx = g'(\tau+t), \quad t \in [0,T]. \end{cases} \tag{5.12}$$

We define the new unknowns

$$v_\tau(t) = v(\tau+t), \quad h_\tau(t) = h(\tau+t), \quad g_\tau(t) = g(\tau+t), \tag{5.13}$$

and we observe that

$$1 * v(\tau+t,x) = \int_0^{\tau+t} v(s)ds = \int_0^\tau \widehat{v}(s)ds + \int_\tau^{\tau+t} v(s)ds$$

$$= \int_0^\tau \widehat{v}(s)ds + \int_0^t v_\tau(s')ds'$$

where we have set $s - \tau = s'$ and defined

$$\widetilde{u}_0(x) := u_0(x) + 1 * \widehat{v}(t,x).$$

So we can rewrite

$$F_u(u_0(x) + 1 * v(\tau + t, x))\, v(\tau + t, x) = F_u(\widetilde{u}_0(x) + 1 * v_\tau(t, x))\, v_\tau(t, x).$$

Let now $0 < t < \tau$. Thanks to the splitting

$$\begin{aligned}
&\int_0^{\tau+t} h(\tau + t - s)\Delta v(s, x)ds \\
&= \int_0^{\tau} \widehat{h}(\tau + t - s)\Delta \widehat{v}(s, x)ds + \int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds \\
&= \int_0^{t} h_\tau(t - s)\Delta \widehat{v}(s, x)ds + \int_t^{\tau} \widehat{h}(\tau + t - s)\Delta \widehat{v}(s, x)ds \\
&\quad + \int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds
\end{aligned} \tag{5.14}$$

and setting $s - \tau = s'$, we have

$$\int_\tau^{t+\tau} \widehat{h}(\tau + t - s)\Delta v(s, x)ds = \int_0^{t} \widehat{h}(t - s')\Delta v_\tau(s', x)ds'.$$

Consequently, the convolution term becomes linear in the unknowns involved in the convolution so that the system becomes:

$$\begin{cases}
D_t v_\tau(t, x) = \Delta v_\tau(t, x) + h_\tau(t)\Delta u_0 + h_\tau * \Delta \widehat{v}(t, x) \\
+\widehat{h} * \Delta v_\tau(t, x) + F_u(\widetilde{u}_0(x) + 1 * v_\tau(t, x))v_\tau(t, x) + \widetilde{F}(t, x) \\
v_\tau(0, x) = u_\tau(x), \quad x \in \Omega, \\
D_\nu v_\tau(t, x) = 0, \quad t \in [0, T],\ x \in \partial\Omega, \\
\Phi[v_\tau(t, \cdot)] := \int_\Omega \phi(x)v_\tau(t, x)dx = g'_\tau(t), \quad t \in [0, T],
\end{cases} \tag{5.15}$$

where we have set

$$\widetilde{F}(t, x) := \int_t^{\tau} \widehat{h}(\tau + t - s)\Delta \widehat{v}(s, x)ds.$$

Steps (g) and (h). *We deduce the a priori estimates for v_τ and h_τ.*
The idea is to get – thanks to the fact that F_u is bounded – the *a priori* estimates for the unknowns v_τ and h_τ. The proof is based on the following lemma.

LEMMA 5.1 *Assume that the assumptions (I1)–(I10) are fulfilled, $p \geq 2$. Let $(\widehat{v}, \widehat{h}) \in X(\tau, p) \times L^p(0, \tau)$ be a solution of (5.4)–(5.6) in $[0, \tau] \times \Omega$, with $0 < \tau < T$. Then, there exists $C > 0$, such that, for all $\delta \in (0, \tau \wedge (T - \tau)]$, if $(v, h) \in X(\tau + \delta, p) \times L^p(0, \tau + \delta)$ is a solution of (5.4)–(5.6) in $[0, \tau + \delta] \times \Omega$, then*

$$\|v_\tau\|_{X(\tau+\delta,p)} + \|h_\tau\|_{L^p(0,\tau+\delta)} \leq C.$$

Proof. It is that of Lemma 7.3 in [7].

Now, in a finite number of steps we can extend the solution to the interval $[0, T]$. For all the details see Section 7 in [7].

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Fourth order ordinary differential operators with general Wentzell boundary conditions [1]

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Abstract
We consider the fourth order ordinary differential operator $Au := (au'')''$ with boundary conditions

$$Au(j) + \beta_j(au'')'(j) + \gamma_j u(j) = 0, \qquad j = 0, 1$$

and one of $u'(j), u''(j)$ vanishes for $j = 0, 1$. Here $\beta_0 < 0 < \beta_1$.
Then A is essentially selfadjoint and bounded below on the Hilbert space $H = L^2(0,1) \oplus C_w^2$, the completion of $C[0,1]$ under the inner product

$$(u,v)_H = \int_0^1 u(x)\overline{v(x)}\,dx + \sum_{j=0}^{1} w_j u(j)\overline{v(j)}$$

where $w_j := (-1)^{j+1}/\beta_j$ for $j = 0, 1$. Applications to partial differential equations are given.

1 Introduction

In some previous papers (see, e.g., [5], [6], [4]) we showed how to solve linear parabolic equations of the form $D_t u = Au$ (A a second order elliptic operator) with boundary conditions of the form $\alpha Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0$ on $\partial\Omega$, provided that $\beta, \gamma \in C^1(\partial\Omega)$, $\beta > 0$ on $\partial\Omega$. Here we find the corresponding results for the fourth order operator A of the type $Au := (au'')''$, where we assume that

$(A1)$ $\qquad a \in C^4[0,1], \quad a(x) > 0 \quad \text{in} \quad [0,1],$

with general Wentzell boundary conditions of the type

$(BC)_j$ $\qquad Au(j) + \beta_j(au'')'(j) + \gamma_j u(j) = 0 \quad j = 0, 1,$

[1]Work partially supported by the GNAMPA-INdAM.

where $\beta_0 < 0 < \beta_1$, and $\gamma_j \in \mathbf{R}$, for $j = 0, 1$. In order to study positivity and essential selfadjointness of A in suitable Hilbert spaces, obtained as completions of $C[0,1]$ with respect to an inner product depending on β_j, $j = 0, 1$, additional boundary conditions must be imposed. Notice that boundedness below and essential selfadjointness of an operator B guarantee the existence of an analytic semigroup and of a cosine function generated by the closure of $-B$ (see, e.g., [10], Theorem 6.12, Theorem 6.6, Theorem 7.4, Theorem 8.5). For the treatment of the higher dimensional case we refer to [7]. A classification of general boundary conditions for symmetry, boundedness below and quasiaccretivity of the operator $Au = u''''$ will be studied in the paper [8]. Examples of fourth order elliptic operators with classical boundary conditions can be found, e.g., in [11], Chapter 2, Section 9.8.

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2 The main results

We consider the case when the coefficient a is sufficiently regular and strictly positive, i.e.,

$$(A1) \qquad a \in C^4[0,1], \quad a(x) > 0 \quad \text{in} \quad [0,1].$$

Then we assume that

$$(A2) \qquad w = (w_1, w_2) \in \mathbf{R}^2, \quad w_j > 0 \quad \text{for} \quad j = 0, 1.$$

$(A3)$ $H := L^2(0,1) \oplus \mathbf{C}^2_w$ denotes the completion of $C[0,1]$ with respect to the norm associated with the inner product

$$(u,v)_H := \int_0^1 u(x)\overline{v(x)}\,dx + \sum_{j=0}^{1} w_j u(j)\overline{v(j)}, \quad u, v \in H.$$

Note that if $u \in H^1(0,1)$, then $u \in C[0,1]$ and u can be identified with $(u, u|_{\{0,1\}}) \in H$.

Let us introduce the following additional boundary conditions:

$$(BC)_2 \qquad u'(0) = 0 = u''(1),$$

$$(BC)_3 \qquad u''(0) = 0 = u'(1),$$

$$(BC)_4 \qquad u'(0) = 0 = u'(1),$$

$$(BC)_5 \qquad u''(0) = 0 = u''(1).$$

Let us consider $u, v \in C^4[0,1]$ which verify $(BC)_j$, where $\beta_0 < 0 < \beta_1$, and $\gamma_j \in \mathbf{R}$, for $j = 0, 1$. We start by studying the symmetry, if A is defined on

$$D_k(A) := \{u \in C^4[0,1] : (BC)_j, \quad j = 0, 1, \quad \text{and} \quad (BC)_k \quad \text{hold}\}$$

for some $k = 2, 3, 4, 5$, where $w_j := (-1)^{j+1}/\beta_j$ for $j = 0, 1$.

Let us evaluate

$$(Au, v)_H = \int_0^1 (au'')''(x)\overline{v(x)}\,dx + \sum_{j=0}^{1} w_j Au(j)\overline{v(j)} \tag{2.1}$$

and denote by

$$C_1 := \sum_{j=0}^{1} w_j Au(j)\overline{v(j)}. \tag{2.2}$$

Integration by parts in (2.1) gives

$$\begin{aligned}
(Au, v)_H &= \int_0^1 (au'')''(x)\overline{v(x)}\,dx + C_1 \qquad (2.3)\\
&= -\int_0^1 (au'')'(x)\overline{v'(x)}\,dx + (au'')'\bar{v}|_0^1 + C_1\\
&= \int_0^1 (au'')(x)\overline{v''(x)}\,dx - au''\bar{v}'|_0^1 + B_1 + C_1\\
&\quad (where \quad B_1 = (au'')'\bar{v}|_0^1)\\
&= -\int_0^1 u'(x)(\overline{av''})'(x)\,dx + u'a\bar{v}''|_0^1 + B_2 + B_1 + C_1\\
&\quad (where \quad B_2 = -(au'')\bar{v}'|_0^1)\\
&= \int_0^1 u(x)(\overline{av''})''(x)\,dx - u\overline{(av'')'}|_0^1 + B_3 + B_2 + B_1 + C_1\\
&\quad (where \quad B_3 = u'(\overline{av''})|_0^1)\\
&= \int_0^1 u(x)\overline{Av(x)}\,dx + \sum_{i=1}^{4} B_i + C_1\\
&\quad (where \quad B_4 = -u\overline{(av'')'}|_0^1)\\
&= (u, Av)_H + \sum_{i=1}^{4} B_i + C_1 - \widetilde{C_1}\\
&\quad (where \quad \widetilde{C_1} = \sum_{j=0}^{1} w_j u(j)\overline{Av(j)})\ .
\end{aligned}$$

We show that if we add one of the additional boundary conditions $(BC)_k$

for $k = 2, 3, 4, 5$, then

$$\sum_{i=1}^{4} B_i + C_1 - \widetilde{C_1} = 0, \tag{2.4}$$

and, consequently, $(A, D_k(A))$ is symmetric.

Let us examine all the cases corresponding to $(BC)_k$, $k = 2, 3, 4, 5$.

Case $(BC)_2$. Assume that u, v satisfy $(BC)_2$, i.e.,

$$u'(0) = 0 = u''(1) \quad and \quad v'(0) = 0 = v''(1).$$

This implies

$$B_2 = B_3 = 0. \tag{2.5}$$

Moreover, from the boundary conditions $(BC)_j$, $j = 0, 1$, it follows that

$$\begin{aligned} Au(j) &= -\beta_j (au'')'(j) - \gamma_j u(j), \quad j = 0, 1 \\ Av(j) &= -\beta_j (av'')'(j) - \gamma_j v(j), \quad j = 0, 1 \ . \end{aligned}$$

Hence

$$\begin{aligned} C_1 - \widetilde{C_1} &= \sum_{j=0}^{1} w_j Au(j)\overline{v(j)} - \sum_{j=0}^{1} w_j u(j)\overline{Av(j)} \\ &= \sum_{j=0}^{1} w_j \overline{v(j)}[-\beta_j (au'')'(j) - \gamma_j u(j)] \\ &\quad - \sum_{j=0}^{1} w_j u(j)[-\beta_j \overline{(av'')'(j)} - \gamma_j \overline{v(j)}] \\ &= \sum_{j=0}^{1} w_j \beta_j [u(j)\overline{(av'')'(j)} - \overline{v(j)}(au'')'(j)]. \end{aligned} \tag{2.6}$$

On the other hand

$$\begin{aligned} B_1 + B_4 &= (au'')'(1)\overline{v(1)} - u(1)\overline{(av'')'(1)} \\ &\quad + u(0)\overline{(av'')'(0)} - (au'')'(0)\overline{v(0)}. \end{aligned} \tag{2.7}$$

Thus, plugging (2.5), (2.6) and (2.7) in the left hand side of (2.4), we obtain

$$\begin{aligned} &(1 - w_1\beta_1) \cdot [(au'')'(1)\overline{v(1)} - u(1)\overline{(av'')'(1)}] \\ &+ (1 + w_0\beta_0) \cdot [u(0)\overline{(av'')'(0)} - (au'')'(0)\overline{v(0)}] = 0, \end{aligned} \tag{2.8}$$

and assertion (2.4) holds.

Case $(BC)_3$. Assume that u, v satisfy $(BC)_3$, i.e.,

$$u''(0) = v''(0) = 0, \quad u'(1) = v'(1) = 0.$$

Thus, as in case $(BC)_2$, $B_2 = B_3 = 0$ and similar arguments as in case $(BC)_2$ allow us to conclude that assertion (2.4) holds.

Case $(BC)_4$. Suppose that u, v satisfy $(BC)_4$, i.e.,

$$u'(0) = v'(0) = 0, \quad u'(1) = v'(1) = 0.$$

Then again $B_2 = B_3 = 0$ and similar arguments as in case $(BC)_2$ lead to the conclusion that assertion (2.4) holds.

Case $(BC)_5$. Assume that u, v satisfy $(BC)_5$, i.e.,

$$u''(0) = v''(0) = 0, \quad u''(1) = v''(1) = 0.$$

Therefore, $B_2 = B_3 = 0$ and as before the assertion follows.

In order to prove positivity, let us consider $u \in C^4[0,1]$ and observe that, according to calculations in (2.3), we have

$$\begin{aligned}(Au, u)_H &= \int_0^1 a|u''|^2\,dx - au''\bar{u}'|_0^1 + (au'')'\bar{u}|_0^1 \\ &\quad + \sum_{j=0}^{1} w_j Au(j)\overline{u(j)}.\end{aligned}$$

If, in addition, u satisfies $(BC)_j$, $j = 0, 1$, and $(BC)_k$ for some $k = 2, 3, 4, 5$, then $au''\bar{u}'|_0^1 = 0$ and we obtain

$$\begin{aligned}(Au, u)_H &= \int_0^1 a|u''|^2\,dx + (au'')'\bar{u}|_0^1 \\ &\quad + \sum_{j=0}^{1} w_j Au(j)\overline{u(j)}.\end{aligned} \tag{2.9}$$

Now, by taking into account $(BC)_j$, $j = 0, 1$, we have

$$\begin{aligned}(Au, u)_H &= \int_0^1 a|u''|^2\,dx + (au'')'\bar{u}|_0^1 \\ &\quad + \sum_{j=0}^{1} w_j[-\beta_j(au'')'(j) - \gamma_j u(j)]\overline{u(j)} \\ &= \int_0^1 a|u''|^2\,dx + (1 - \beta_1 w_1)(au'')'(1)\overline{u(1)} \\ &\quad - (1 + \beta_0 w_0)(au'')'(0)\overline{u(0)} - \gamma_0 w_0|u(0)|^2 - \gamma_1 w_1|u(1)|^2.\end{aligned} \tag{2.10}$$

This is the reason we assume $w_j := (-1)^{j+1}/\beta_j$, $j = 0, 1$. Then this choice of (w_0, w_1) uniquely determines H. It follows that

$$(Au, u)_H = \int_0^1 a|u''|^2\,dx - \gamma_0 w_0 |u(0)|^2 - \gamma_1 w_1 |u(1)|^2. \tag{2.11}$$

Thus, if we add the assumption $\gamma_j \le 0$, $j = 0, 1$, it yields that

$$(Au, u)_H = \int_0^1 a|u''|^2\,dx + \sum_{j=0}^{1} |\gamma_j| w_j |u(j)|^2 \ge 0. \tag{2.12}$$

In particular, if $\gamma_j < 0$, $j = 0, 1$, then

$$(Au, u)_H = \int_0^1 a|u''|^2\,dx + \sum_{j=0}^{1} |\gamma_j| w_j |u(j)|^2 > 0, \tag{2.13}$$

unless $u = 0$.

Now we prove the following result which is of independent interest.

LEMMA 2.1 *Given $w_0 > 0, w_1 > 0$, there exists $\varepsilon > 0$ such that for any $u \in C^2[0,1]$ we have*

$$\int_0^1 |u''(x)|^2\,dx + \sum_{j=0}^{1} w_j |u(j)|^2 \ge \varepsilon\Big(\int_0^1 |u(x)|^2\,dx + \sum_{j=0}^{1} w_j |u(j)|^2\Big).$$

Proof. Let us consider $u \in C^2[0,1]$ and $x_0 \in [0,1]$ such that

$$u(1) - u(0) = \int_0^1 u'(x)\,dx = u'(x_0).$$

Then

$$\begin{aligned} u(x) &= \int_0^x u'(y)\,dy + u(0) \\ &= \int_0^x \Big(\int_{x_0}^y u''(z)\,dz + u'(x_0)\Big)dy + u(0) \\ &= \int_0^x \Big(\int_{x_0}^y u''(z)\,dz\Big)\,dy + xu'(x_0) + u(0). \end{aligned}$$

Since $(a+b)^2 \le 2a^2 + 2b^2$ for any $a, b \in \mathbf{R}$, we deduce that

$$\int_0^1 |u(x)|^2\,dx \le 2\int_0^1 |u''(x)|^2\,dx + 2\int_0^1 |xu'(x_0) + u(0)|^2\,dx.$$

Therefore, for any $x \in [0,1]$:

$$\begin{aligned}|xu'(x_0)+u(0)|^2 &\le 2|u'(x_0)|^2+2|u(0)|^2\\ &= 2(|u(1)-u(0)|^2)+2|u(0)|^2\\ &\le 4|u(1)|^2+4|u(0)|^2+2|u(0)|^2\\ &\le 6(|u(1)|^2+|u(0)|^2).\end{aligned}$$

This gives

$$\int_0^1 |u(x)|^2\,dx \le 2\int_0^1 |u''(x)|^2\,dx + 12\sum_{j=0}^1 |u(j)|^2,$$

or, equivalently,

$$\int_0^1 |u''(x)|^2\,dx \ge \frac{1}{2}\int_0^1 |u(x)|^2\,dx - 6\sum_{j=0}^1 |u(j)|^2.$$

It follows that for any $0<\alpha\le 1$, we have

$$\int_0^1 |u''(x)|^2\,dx + \sum_{j=0}^1 w_j|u(j)|^2 \ge \frac{\alpha}{2}\int_0^1 |u(x)|^2\,dx + \sum_{j=0}^1 |u(j)|^2(w_j-6\alpha).$$

Choose $\alpha := \min\{1, w_0/12, w_1/12\}$, thus $w_j - 6\alpha \ge w_j/2$, for $j=0,1$. Hence

$$\int_0^1 |u''(x)|^2\,dx + \sum_{j=0}^1 w_j|u(j)|^2 \ge \frac{\alpha}{2}\Big(\int_0^1 |u(x)|^2\,dx + \sum_{j=0}^1 |u(j)|^2 w_j\Big).$$

Then the assertion is true with $\varepsilon = \alpha/2$.

Let us state our main result.

THEOREM 2.1 *Under the assumptions* $(A1)--(A3)$, *the operator* A *with domain*

$$D_k(A) := \{u \in C^4[0,1] : u \quad \textit{satisfies} \quad (BC)_j, \quad j=0,1, \quad \textit{and} \quad (BC)_k\}$$

is essentially selfadjoint and bounded below on the space H *for any* $k=2,3,4,5$, *provided that* $\beta_0<0<\beta_1$, $w_j := (-1)^{j+1}/\beta_j$, *and* H *is the corresponding Hilbert space. In addition, we have* $A \ge \varepsilon I$ *and* $\varepsilon \ge 0$ *(resp.* $\varepsilon > 0$*) if* $\gamma_j \le 0$ *(resp.* $\gamma_j < 0$*), for* $j=0,1$.

Proof. Let us define A_k the realization of A in H with domain $D_k(A)$ for $k=2,3,4,5$. If $u \in D_k(A)$, then by (2.10) we obtain

$$(A_k u, u)_H = \int_0^1 a|u''|^2\,dx - \sum_{j=0}^1 \gamma_j w_j |u(j)|^2.$$

Thus, if $\gamma_j \le 0$, $j = 0, 1$, then $A_k \ge 0$. In general, we have

$$\begin{aligned}(A_k u, u)_H &\ge a_0 \int_0^1 |u''|^2\,dx - \sum_{j=0}^{1} \gamma_j w_j |u(j)|^2 \\ &\ge -\sum_{j=0}^{1} \gamma_j w_j |u(j)|^2 - \int_0^1 |u|^2\,dx \\ &\ge \min\{-\gamma_0, -\gamma_1, -1\}\|u\|_H^2.\end{aligned}$$

This yields that A_k is bounded below. Moreover, if $\gamma_j < 0$, for $j = 0, 1$, then Lemma 1 allows us to find an $\varepsilon_0 > 0$ such that

$$(A_k u, u)_H \ge \varepsilon_0 \Big(\int_0^1 |u|^2\,dx + \sum_{j=0}^{1} w_j |u(j)|^2 \Big) = \varepsilon_0 \|u\|_H^2.$$

Thus the second assertion of the theorem holds. Now, according to previous calculations, we already know that A_k is symmetric for any $k = 2, 3, 4, 5$. In order to prove that A_k is essentially selfadjoint, it suffices to show that the range of $\lambda I + A_k$ is dense for sufficiently large real λ. To this end, let us consider for each $h \in C^2[0,1]$ the equation

$$\lambda u + A_k u = h \quad \text{in} \quad [0,1]. \tag{2.14}$$

We seek a solution $u \in D_k(A)$ which satisfies (2.14). From $(BC)_j$ and (2.14) we deduce that

$$-\beta_j (a u'')'(j) + (\lambda - \gamma_j) u(j) = h(j), \quad j = 0, 1. \tag{2.15}$$

We begin by finding a weak solution of (2.14). Let $v \in C^2[0,1]$, multiply (2.14) by $\overline{v}$ and integrate to get

$$\lambda \int_0^1 u\overline{v}\,dx + \int_0^1 (a u'')''\overline{v}\,dx = \int_0^1 h\overline{v}\,dx. \tag{2.16}$$

Integration by parts gives

$$\begin{aligned}\lambda \int_0^1 u\overline{v}\,dx + (a u'')'\overline{v}\big|_0^1 - \int_0^1 (a u'')'\overline{v}'\,dx \\ = \int_0^1 h\overline{v}\,dx.\end{aligned} \tag{2.17}$$

From (2.15) we deduce that

$$(a u'')'(j) = \frac{(\lambda - \gamma_j) u(j) - h(j)}{\beta_j}, \quad j = 0, 1,$$

and we obtain that (2.17) becomes

$$\lambda \int_0^1 u\overline{v}\,dx + \frac{(\lambda-\gamma_1)u(1)\overline{v}(1)}{\beta_1} - \frac{(\lambda-\gamma_0)u(0)\overline{v}(0)}{\beta_0} - \int_0^1 (au'')'\overline{v}'\,dx$$
$$= \int_0^1 h\overline{v}\,dx + \frac{h(1)\overline{v}(1)}{\beta_1} - \frac{h(0)\overline{v}(0)}{\beta_0}. \tag{2.18}$$

Again integrating by parts gives

$$\lambda \int_0^1 u\overline{v}\,dx + \frac{(\lambda-\gamma_1)u(1)\overline{v}(1)}{\beta_1} - \frac{(\lambda-\gamma_0)u(0)\overline{v}(0)}{\beta_0}$$
$$-(au'')\overline{v}'|_0^1 + \int_0^1 au''\overline{v}''\,dx$$
$$= \int_0^1 h\overline{v}\,dx + \frac{h(1)\overline{v}(1)}{\beta_1} - \frac{h(0)\overline{v}(0)}{\beta_0}. \tag{2.19}$$

Now, for any $k = 2, 3, 4, 5$, let us introduce

$$V_k := \{u \in C^4[0,1] : u \quad \text{satisfies} \quad (BC)_j, \quad j = 0, 1 \quad \text{and} \quad (BC)_k\},$$

and observe that for any $k = 2, 3, 4, 5$ and any $u, v \in V_k$, the equality (2.19) reduces to

$$\lambda \int_0^1 u\overline{v}\,dx + \frac{(\lambda-\gamma_1)u(1)\overline{v}(1)}{\beta_1} - \frac{(\lambda-\gamma_0)u(0)\overline{v}(0)}{\beta_0} + \int_0^1 au''\overline{v}''\,dx$$
$$= \int_0^1 h\overline{v}\,dx + \frac{h(1)\overline{v}(1)}{\beta_1} - \frac{h(0)\overline{v}(0)}{\beta_0}. \tag{2.20}$$

For $k = 2, 3, 4, 5$ let us denote by K_k the completion of V_k with respect to the norm $\|\cdot\|_K$ given by

$$\|u\|_K := \left(\|u\|_H^2 + \|u''\|^2_{L^2((0,1),a\,dx)}\right)^{1/2}.$$

Let $L(u, v)$ be the left-hand side of (2.20) and let $F(v)$ be the corresponding right-hand side. Thus L is a bounded sesquilinear form on K_k and F is a bounded conjugate linear functional on K_k: indeed for any $u, v \in K_k$ we have

$$|L(u,v)| \leq \max\{|\lambda|, 1\}\|u\|_{K_k}\|v\|_{K_k} + (|\lambda| + \max\{|\gamma_0|, |\gamma_1|\})\,\|u\|_H\|v\|_H$$
$$\leq c_1(\lambda)\|u\|_{K_k}\,\|v\|_{K_k}$$

and

$$|F(v)| \leq \|h\|_H\,\|v\|_{K_k},$$

provided that $h \in H$. Also

$$\operatorname{Re} L(u,u) \geq \min\{\lambda, \lambda-\gamma_1, \lambda-\gamma_0, 1\}\|u\|^2_{K_k}.$$

By the Lax-Milgram lemma, for any $k = 2, 3, 4, 5$, for any $\lambda > \max\{\gamma_1, \gamma_0, 1\}$ and for any $h \in H$ there is a unique $u \in K_k$ such that

$$L(u, v) = F(v), \quad v \in K_k.$$

That is, (2.20) holds and this $u \in K_k$ is our weak solution of (2.14) which satisfies $(BC)_k$, if sufficiently regular. For h in a dense set, we want to show that our weak solution is in $D_k(A)$. If $h \in C^{4+\epsilon}[0, 1]$, then we know that $u \in H^1(0, 1)$ satisfies (in the weak sense)

$$\lambda u + (au'')'' = h \in C^{4+\epsilon}[0, 1],$$

together with the boundary conditions in V_k, when sufficiently regular. Moreover u satisfies the uniformly elliptic problem

$$\lambda v + (av'')'' = h \quad \text{in} \quad (0, 1), \tag{2.21}$$

$$v(j) = \tau_1(j), \quad v'(j) = \tau_2(j), \quad j = 0, 1, \tag{2.22}$$

where $\tau_1(j) = u(j)$, $\tau_2(j) = u'(j)$, for $j = 0, 1$. This implies that $v = u \in H^2(0, 1)$. Next, if we define $z := au''$, it satisfies

$$z'' = h - \lambda v \in H^2(0, 1)$$

and

$$z'(j) = \tau_3(j), \quad j = 0, 1,$$

where $\tau_3(j) = \beta_j^{-1}(\lambda v(j) - \gamma_j v(j) - h(j))$, for $j = 0, 1$. Then $v \in H^4(0, 1)$ and so $u \in C^{3+\delta}[0, 1]$. This implies $z'' \in H^4(0, 1)$ and we obtain that $z \in H^6(0, 1)$. Then by Sobolev's embedding theorems (see, e.g., [3], [9], [12]), $u \in C^4[0, 1]$ and, as u belongs to K_k, it satisfies $(BC)_j$ for $j = 0, 1$. This yields $u \in D_k(A)$. Hence A_k is essentially selfadjoint. This shows our assertion for any $k = 2, 3, 4, 5$.

REMARK 2.1 Notice that in the previous theorem everything works provided that $a \in H^3(0, 1)$. Indeed, for the symmetry of A_k it suffices that $a \in C^2[0, 1]$, while for the range condition of the closure of A_k it suffices (see the last four lines of the proof) that $u'' \in H^3(0, 1)$, what follows from $a \in H^3(0, 1)$. In addition, if we denote by $D := D_x$, then similar arguments as before can work also for B^2, with $B := D(aD)$, and operators of the type $D^2(aD^2) + D(bD) + cI$ acting on H, provided that a satisfies $(A1)$, $b \in C^3[0, 1]$, $c \in C^2[0, 1]$, and suitable additional boundary conditions are considered. Extensions to dimension N work well provided that D is replaced by ∇ (see [7]).

REMARK 2.2 Let $Au := u''''$ on an interval $(\alpha, \beta) \subset\subset \mathbf{R}$. Associate any linear boundary condition with A (e.g., general Wentzell boundary conditions or Robin boundary conditions). We have

$$C_c^\infty(\alpha, \beta) \subset D(A).$$

We know that, in many cases, A is accretive on L^2 (or H), i.e., $-A$ is dissipative. We show that A is *not* quasi-accretive on $C[\alpha, \beta]$. For convenience we take $\varepsilon > 0$ and $[\alpha, \beta] = [-2\varepsilon, 2\varepsilon]$. Assume that $\varepsilon < 1/2$, and take n be an even positive integer sufficiently large, in such a way that

$$(2\varepsilon)^{n-4} \leq \frac{4}{n}.$$

If we define

$$\widetilde{u}(x) = x^n - x^4 + b,$$

with $n \geq 10$, then it follows that $\widetilde{u}$ is even and positive in $[-2\varepsilon, 2\varepsilon]$, provided that $(2\varepsilon)^4 < b$.

Moreover we obtain

$$\widetilde{u}''''(x) = -24 + n(n-1)(n-2)(n-3)x^{n-4}.$$

Let $u := \widetilde{u}\varphi$, where $\varphi \in C_c^\infty(-2\varepsilon, 2\varepsilon)$, φ even, $\varphi \equiv 1$ in $[0, \varepsilon]$, and decreasing in $(\varepsilon, 2\varepsilon)$. Hence $u \in D(A)$ (no matter which boundary conditions we use) and $u^{(\tau)} = \widetilde{u}^{(\tau)}$ on $[-\varepsilon, \varepsilon]$, for any $\tau \in \mathbf{N}$.

Notice that $\widetilde{u}$ is even and we have

$$\widetilde{u}'(x) = nx^{n-1} - 4x^3 = 4x^3\left(\frac{n}{4}x^{n-4} - 1\right) < 0 \quad \text{on} \quad [0, 2\varepsilon].$$

Therefore $\widetilde{u}$ is decreasing in $[0, 2\varepsilon]$ and $\max\limits_{x \in [-2\varepsilon, 2\varepsilon]} \widetilde{u}(x) = \widetilde{u}(0) = b$. Since we have $u' = \widetilde{u}'\varphi + \widetilde{u}\varphi'$, it follows that u is decreasing in $[0, 2\varepsilon]$ and

$$\|u\|_{C[-2\varepsilon, 2\varepsilon]} = b = u(0).$$

In addition we have

$$n(n-1)(n-2)(n-3)n^{4-n} < 1.$$

Hence

$$u''''(0) = Au(0) \in [-24, -23].$$

On $C[-2\varepsilon, 2\varepsilon] = C[\alpha, \beta]$, we consider the mapping

$$\psi := b\delta_0 = \|u\|_\infty \delta_0 \in J(u),$$

where J is the duality map and $\delta_0(u) := u(0)$ (see, e.g., [2], Chapter II Examples 3.26). Observe that

$$< Au, \psi >= bu''''(0) \in [-24\,b, -23\,b].$$

Consequently $< Au, \psi >$ cannot be nonnegative for all $\psi \in J(u)$.

Does exist $\omega \in \mathbf{R}$ such that $A + \omega I$ is accretive?

The answer is no. Indeed, if it were yes, then we should obtain

$$\begin{aligned} < Au + \omega u, \psi > &= bu''''(0) + \omega \|u\|_\infty^2 \\ &= bu''''(0) + \omega b^2 \\ &= b(u''''(0) + \omega b) \\ &< (-23 + \omega b)b. \end{aligned}$$

Thus, given $\omega > 0$, if we choose $b \in (0, 23/\omega)$, we should get the contradiction

$$< Au + \omega u, \psi > < 0.$$

We worked on $C[-2\varepsilon, 2\varepsilon]$, but we could rescale to make things work on $[0, 1]$. For other relations between boundary conditions and accretivity when $Au = u''''$ see [7].

REMARK 2.3 Let us consider the operator $A_1 u := (au'')''$ on $C^4[0, 1]$, where

$$a \in C^4[0, 1], \quad a(x) > 0 \quad \text{for all} \quad x \in [0, 1].$$

We equip A_1 with general Wentzell boundary conditions $(BC)_j$, for $j = 0, 1$, where $\gamma_j \in \mathbf{R}$, $\beta_0 < 0 < \beta_1$, and with $(BC)_k$, for $k = 2, 3, 4, 5$. Then A_1 is essentially selfadjoint and $A_1 \geq \varepsilon I$ on H. In addition, $\varepsilon \geq 0$ if $\gamma_0, \gamma_1 \leq 0$ and $\varepsilon > 0$ if $\gamma_0, \gamma_1 < 0$. Also, for $a \equiv 1$, let us consider the operator $B := D_x^2$ on $C^2[0, 1]$. It is essentially selfadjoint in H if the boundary conditions are

$$Bu(j) + \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1.$$

Then $A_2 := B^2$ on H has its boundary conditions

$$u''''(j) + \beta_j u'''(j) + \gamma_j u''(j) = 0, \quad j = 0, 1 \tag{2.23}$$

$$u''(j) + \beta_j u'(j) + \gamma_j u(j) = 0, \quad j = 0, 1. \tag{2.24}$$

All of these operators, i.e., A_1 for $a \equiv 1$ with $(BC)_j$, $j = 0, 1$ and $(BC)_k$ for $2 \leq k \leq 5$, and $A_2 = B^2$ with (2.23) − −(2.24) agree on the same domain

$$C_0^4(0, 1) := \{u \in C^4[0, 1] \ : \ u^{(\tau)}(j) = 0, \ 0 \leq \tau \leq 4, \ j = 0, 1\}.$$

Moreover, for any of these $A's$ we have

$$\dim D(A)/C_0^4(0, 1) \ < \ \infty$$

(see [7], Appendix). Thus, if $\lambda \in \rho(A_1) \cap \rho(A_2)$, then

$$(\lambda - A_1)^{-1} - (\lambda - A_2)^{-1}$$

is a finite rank operator. Since $A_2 = B^2$ has a compact resolvent (since B does by [1]), so do all of our A_k.

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Study of elliptic differential equations in UMD spaces [1]

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Abstract Some new results on complete abstract second-order equations of elliptic type in a UMD Banach space are described.
Invoking properties of operators with bounded imaginary powers and using the celebrated Dore-Venni Theorem on the sum of two closed linear operators, existence, uniqueness and maximal L^p regularity of the strict solution are proved.
Some applications to partial differential equations are indicated.
This work completes the results obtained very recently in the framework of Hölder-continuous functions.

1 Introduction and hypotheses

Let us consider, in the complex Banach space X, the abstract differential equation of the second order

$$u''(x) + 2Bu'(x) + Au(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

together with the boundary conditions

$$u(0) = u_0, \quad u(1) = u_1. \tag{1.2}$$

Here, A, B are two closed linear operators in X with domains $D(A)$ and $D(B)$, respectively, $f \in L^p(0,1;X)$, $1 < p < \infty$ and u_0, u_1 are given elements in X. We seek for a strict solution u to (1.1)–(1.2), i.e., a function u such that

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), \quad u' \in L^p(0,1;D(B)),$$

and satisfying (1.1) and (1.2).

Generally, more regularity is required for f to obtain a strict solution, unless X has some particular geometrical properties. This is why we assume in all

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this study that

$$X \text{ is a UMD space.} \tag{1.3}$$

We recall that a Banach space X is a UMD space if and only if for some $p > 1$ (and thus for all p) the Hilbert transform is continuous from $L^p(\mathbb{R}; X)$ into itself (see Bourgain [1], Burkholder [2]).

Moreover we suppose that

$$\begin{cases} B^2 - A \text{ is a linear closed densely defined operator in } X, \\ \mathbb{R}_- \subset \rho(B^2 - A) \text{ and } \exists C > 0 : \forall \lambda \geqslant 0, \\ \left\|(\lambda I + B^2 - A)^{-1}\right\|_{L(X)} \leqslant C/(1+\lambda), \end{cases} \tag{1.4}$$

(it is well known that hypothesis (1.4) implies that $-(B^2 - A)^{1/2}$ is the infinitesimal generator of an analytic semigroup X), ([11], p. 119)

$$D(A) \subseteq D(B^2), \tag{1.5}$$

$$B(B^2 - A)^{-1} y = (B^2 - A)^{-1} B y, \ \forall y \in D(B), \tag{1.6}$$

$$D((B^2 - A)^{1/2}) \subseteq D(B). \tag{1.7}$$

Under these hypotheses, we will study (1.1)–(1.2) in the two following cases

1. first case:

$$B \text{ generates a strongly continuous group } \left(e^{xB}\right)_{x \in \mathbb{R}} \text{ on } X, \tag{1.8}$$

and

$$\begin{cases} \forall s \in \mathbb{R}, \ (B^2 - A)^{is} \in L(X) \text{ and} \\ \exists C \geqslant 1, \ \theta_0 \in]0, \pi[: \ \forall s \in \mathbb{R}, \ \left\|(B^2 - A)^{is}\right\| \leqslant C e^{\theta_0 |s|}, \end{cases} \tag{1.9}$$

one writes $B^2 - A \in BIP(\theta_0, X)$ (bounded imaginary powers).

2. second case:

$$A \text{ is boundedly invertible,} \tag{1.10}$$

$$D(BA) \subset D(B^3), \tag{1.11}$$

$$\pm B - (B^2 - A)^{1/2} \text{ generates an analytic semigroup on } X, \tag{1.12}$$

and

$$\begin{cases} \forall s \in \mathbb{R}, \ (\pm B + (B^2 - A)^{1/2})^{is} \in L(X) \text{ and} \\ \exists C \geqslant 1, \ \theta_\pm \in]0, \pi/2[: \\ \forall s \in \mathbb{R}, \ \left\|\left(\pm B + (B^2 - A)^{1/2}\right)^{is}\right\| \leqslant C e^{\theta_\pm |s|}. \end{cases} \tag{1.13}$$

In the last decades many researchers focused their attention to the resolution of (1.1)–(1.2), when X is any complex Banach space and f is a given X-valued function whose regularity in x depends on the functional ambient, usually

$$f \in C^\theta([0,1];X) \text{ or } f \in W^{\theta,p}(0,1;X), \quad 0<\theta<1,\ 1<p<\infty.$$

A very extensive study of (1.1)–(1.2), with $B \equiv 0$ even with more general boundary conditions can be found in Krein [11]. Other approaches, always concerning $B \equiv 0$, are used in the famous Da Prato-Grisvard paper [3] on the sum of linear operators. Such a method yields interesting results by Labbas-Terreni [12], [13], on more complicated situations, for instance, the case of variable operator coefficients $A(x)$ and $B \equiv 0$.

The case $B \neq 0$ seems more difficult to be handled. Very interesting approaches to (1.1)–(1.2), where A is even substituted by $A + \lambda I$, with λ a complex parameter, are described in the recent monograph by S. Yakubov and Y. Yakubov [21]. They have worked in a Hilbert space H, $-A$ is supposed to be a positive operator in H, $D(A)$ being compactly embedded into H, and B is a closed linear operator in H satisfying at least a condition like

$$\forall \varepsilon > 0,\ \exists C(\varepsilon) > 0 : \forall u \in D(A) \quad \|Bu\|_H \leqslant \varepsilon \|u\|_{(D(A),H)_{1/2,1}} + C(\varepsilon)\|u\|_H .$$

Here, we recall that for all $\theta \in]0,1[$ and $p \in [1,\infty]$, $(D(A),H)_{\theta,p}$ is the well known real interpolation space, see Lions-Peetre [14].

Extending the case when A and B are two scalars, in last years, Labbas and co-authors have considered (1.1)–(1.2) under the expected positivity assumption

$$\begin{cases} B^2 - A \text{ is a linear closed operator in } X \\ \mathbb{R}_- \subset \rho(B^2 - A) \text{ and } \exists C > 0 : \forall \lambda \geqslant 0 \\ \left\|(\lambda I + B^2 - A)^{-1}\right\|_{L(X)} \leqslant C/(1+\lambda), \end{cases} \tag{1.14}$$

where the domain $D(B^2 - A)$ may be not dense. More precisely El Haial and Labbas [6] proved that if (1.14), (1.6) and (1.8) are satisfied, together with some resolvent estimates, then (1.1)–(1.2) has a unique strict solution fulfilling

$$u \in C^2([0,1];X) \cap C([0,1];D(A)),\ u' \in C([0,1];D(B)),$$

provided that f is Hölder continuous, u_0, u_1 belong to a suitable subspace in X and verify conditions of compatibility with respect to equation (1.1).

More recently, by using a quite different approach, extending the semigroup techniques by Krein [11], Favini, Labbas, Tanabe, Yagi [8] have proved that if (1.4)∼(1.8) hold, then (1.1)–(1.2) has a unique strict solution for f Hölder continuous, $u_0, u_1 \in D(A)$.

To avoid the group assumption (1.8), in a very recent paper, Favini, Labbas, Maingot, Tanabe and Yagi [7] show that if (1.4)∼(1.7) and (1.10)∼(1.12) hold then problem (1.1)–(1.2) has a unique strict solution for $f \in C^{\theta}([0,1];X)$, $0<\theta<1$, $u_0, u_1 \in D(A)$. Moreover, if

$$f(i), Au_i \in (D(A),X)_{(2-\theta)/2,\infty}, \quad i=0,1,$$

then u has the maximal regularity property $u'', Bu', Au \in C^{\theta}([0,1];X)$.

Here, we study the case $f \in L^p(0,1;X), 1<p<\infty$, X being a UMD Banach space, using the representation formula of the solution given in Favini, Labbas, Maingot, Tanabe and Yagi [7].

The main new result in this paper, see Theorem 4.2, affirms that under assumptions (1.3)∼(1.7) and (1.10)∼(1.13) problem (1.1)–(1.2) has a unique strict solution in $L^p(0,1;X)$ provided that $u_0, u_1 \in (D(A),X)_{1/(2p),p}$.

Our techniques are based upon the celebrated Dore-Venni Theorem [4] on the sum of two closed linear operators and on the reiteration Theorem in interpolation theory, [14], [20].

Let us give some remarks about our assumptions

REMARK 1.1

1. If we assume (1.3) then X is reflexive, hence (1.4) is equivalent to (1.14); see Haase [9], proposition 1.1. statement h), p. 18–19.

2. It has been shown in Favini, Labbas, Maingot, Tanabe and Yagi [7] that if we assume (1.4)∼(1.7), and (1.10) then

$$\begin{aligned}(1.11) &\iff B+(B^2-A)^{1/2} \text{ is boundedly invertible}\\ &\iff B-(B^2-A)^{1/2} \text{ is boundedly invertible.}\end{aligned}$$

   Moreover in this case, we have

$$\begin{cases} \left(B-(B^2-A)^{1/2}\right)^{-1} = \left(B+(B^2-A)^{1/2}\right)A^{-1} \\ \left(B+(B^2-A)^{1/2}\right)^{-1} = \left(B-(B^2-A)^{1/2}\right)A^{-1}. \end{cases} \tag{1.15}$$

3. From (1.8) we deduce that B^2 generates a bounded holomorphic semigroup in X (see Stone [19]) and if we assume (1.4)∼(1.6) together with (1.8), the Da Prato-Grisvard sum's theory [3], applied to operators $-(B^2-A)$ and B^2, gives

$$(A-\lambda I)^{-1} \in L(X), \quad \left\|(A-\lambda I)^{-1}\right\|_{L(X)} \leqslant K/(1+\lambda),$$

   for all $\lambda \geqslant 0$. In particular (1.10) is fulfilled.

4. Under assumptions (1.4)∼(1.8) and if in addition we assume (1.11) then (1.12) is satisfied and

$$e^{-x[B+(B^2-A)^{1/2}]} = e^{-xB}e^{-x(B^2-A)^{1/2}}, \quad e^{x[B-(B^2-A)^{1/2}]} = e^{xB}e^{-x(B^2-A)^{1/2}},$$

   see Favini, Labbas, Tanabe and Yagi [8].

The plan of the paper is as follows.

In Section 2 we recall the representation formula of the solution u.

Section 3 is devoted to the case $B = 0$ which is a good model to clarify the techniques used in this study. Theorem 3.1 furnishes an extension of S. Yakubov and Y. Yakubov [21], p. 291–292 (when in fact a complex parameter is added to A, too), from Hilbert spaces to UMD spaces.

Section 4 completes our work in the general case. In a first approach we assume that B generates a group. In a second approach, in order to avoid this group assumption we suppose that $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup.

Finally in Section 5 we give some examples of application to partial differential equations.

2 Representation of the solution

We assume here (1.3)∼(1.7) and (1.10)∼(1.12). Let us denote $(T_0(x))_{x\geqslant 0}$ and $(T_1(x))_{x\geqslant 0}$ the analytic semigroups generated by

$$-B - (B^2 - A)^{1/2} \quad \text{and} \quad B - (B^2 - A)^{1/2}.$$

Then a representation formula of the solution of Problem (1.1)–(1.2) is given by

$$\begin{aligned} u(x) = T_0(x)\xi_0 + T_1(1-x)\xi_1 & \\ -\frac{1}{2}(B^2 - A)^{-1/2}\int_0^x T_0(x-s)f(s)ds & \qquad (2.1) \\ -\frac{1}{2}(B^2 - A)^{-1/2}\int_x^1 T_1(s-x)f(s)ds, & \end{aligned}$$

for $x \in (0, 1)$, where

$$\begin{aligned} \xi_0 = (I - Z)^{-1}(u_0 - T_1(1)u_1) & \\ +\frac{1}{2}(I - Z)^{-1}(B^2 - A)^{-1/2}\int_0^1 T_1(s)f(s)ds & \qquad (2.2) \\ -\frac{1}{2}(I - Z)^{-1}(B^2 - A)^{-1/2}T_1(1)\int_0^1 T_0(1-s)f(s)ds, & \end{aligned}$$

$$\begin{aligned} \xi_1 = (I - Z)^{-1}(u_1 - T_0(1)u_0) & \\ +\frac{1}{2}(I - Z)^{-1}(B^2 - A)^{-1/2}\int_0^1 T_0(1-s)f(s)ds & \qquad (2.3) \\ -\frac{1}{2}(I - Z)^{-1}(B^2 - A)^{-1/2}T_0(1)\int_0^1 T_1(s)f(s)ds, & \end{aligned}$$

and

$$Z = e^{-2(B^2-A)^{1/2}},$$

(see [7]). Notice that since the imaginary axis is contained in the resolvent set $\rho\left(-(B^2-A)^{1/2}\right)$, $I-Z$ has a bounded inverse (see Lunardi [15], p. 60).

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3 Study in the case $B \equiv 0$

In this case, our previous Problem becomes

$$\begin{cases} u''(x) + Au(x) = f(x), & x \in (0,1), \\ u(0) = u_0, \ u(1) = u_1. \end{cases} \tag{3.1}$$

Assumptions (1.4)∼(1.9) reduce to

$$\begin{cases} A \text{ is a linear closed densely defined operator in } X, \, \mathbb{R}_+ \subset \rho(A) \\ \text{and } \exists C \geqslant 1 : \forall \lambda \geqslant 0, \ \left\| (\lambda I - A)^{-1} \right\|_{L(X)} \leqslant C/(1+\lambda), \end{cases} \tag{3.2}$$

and

$$\exists C \geqslant 1, \theta \in]0, \ \pi[\ : \ \forall s \in \mathbb{R}, \ \left\| (-A)^{is} \right\| \leqslant C e^{\theta |s|}. \tag{3.3}$$

REMARK 3.1 Assume (3.2). Then

1. $-\sqrt{-A}$ generates an analytic semigroup $\left(e^{-\sqrt{-A}x}\right)_{x \geqslant 0}$ in X.
2. For any $\beta \in \mathbb{C}$
$$((-A)^{1/2})^{\beta} = (-A)^{\beta/2}$$
(see Haase [9], Proposition 2.18, statement e, p. 64) from which we deduce that (3.3) is equivalent to
$$\exists C \geqslant 1, \ \theta \in]0, \ \pi[\ : \ \forall s \in \mathbb{R}, \ \left\| \left(\sqrt{-A}\right)^{is} \right\| \leqslant C e^{(\theta/2)|s|}.$$

Due to assumptions (1.3), (3.2), (3.3) and the previous remark, statement 2, the Dore-Venni Theorem [4] gives directly, for $g \in L^p(0,1;X)$,

$$x \longmapsto L(x,g) = \sqrt{-A} \int_0^x e^{-(x-s)\sqrt{-A}} g(s) ds \in L^p(0,1;X), \tag{3.4}$$

and consequently

$$x \longmapsto \mathcal{L}(x,g) = \sqrt{-A} \int_0^1 e^{-(x+s)\sqrt{-A}} g(s) ds \in L^p(0,1;X), \tag{3.5}$$

since

$$\mathcal{L}(x,g) = L(x,g_1) + e^{-2x\sqrt{-A}}L(1-x,g(1-.)),$$

where $g_1(s) = e^{-2s\sqrt{-A}}g(s)$.

We also have the following Lemma.

LEMMA 3.1 *Assume (3.2). Then, for any* $w \in (D(A),X)_{1/(2p),p}$

$$x \longmapsto M(x,w) = Ae^{-x\sqrt{-A}}w$$

belongs to $L^p(0,1;X)$.

Proof. If $w \in (D(A),X)_{1/(2p),p}$, then

$$\begin{aligned}\int_0^1 \left\|Ae^{-x\sqrt{-A}}w\right\|^p dx &= \int_0^1 x^{2\cdot\frac{1}{2p}\cdot p}\left\|\left(-\sqrt{-A}\right)^2 e^{-x\sqrt{-A}}w\right\|^p \frac{dx}{x} \\ &\leqslant \int_0^{+\infty} x^{2\cdot\frac{1}{2p}\cdot p}\left\|\left(-\sqrt{-A}\right)^2 e^{-x\sqrt{-A}}w\right\|^p \frac{dx}{x} \\ &\leqslant C\left\|w\right\|_{(D(A),X)_{1/(2p),p}},\end{aligned}$$

in view of Lions-Peetre Theorem, see [14].

The main result in this section is

THEOREM 3.1 *Assume (1.3), (3.2) and (3.3). If* $f \in L^p(0,1;X)$ *with* $1 < p < \infty$, *and* $u_0, u_1 \in (D(A),X)_{1/(2p),p}$, *then Problem (3.1) has a unique strict solution* u, *that is*

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)),$$

and satisfies (3.1).

Proof. We can suppose, without loss of generality, that $u_1 = 0$. The representation formula (2.1) reduces to

$$\begin{aligned}u(x) &= e^{-x\sqrt{-A}}\xi_0 + e^{-(1-x)\sqrt{-A}}\xi_1 - \frac{1}{2}(-A)^{-1/2}\int_0^x e^{-(x-s)\sqrt{-A}}f(s)ds \\ &\quad -\frac{1}{2}(-A)^{-1/2}\int_x^1 e^{-(s-x)\sqrt{-A}}f(s)ds,\end{aligned}$$

with

$$\begin{aligned}\xi_0 = (I-Z)^{-1}\Big\{u_0 &+ \frac{1}{2}(-A)^{-1/2}\int_0^1 e^{-s\sqrt{-A}}f(s)ds \\ &-\frac{1}{2}(-A)^{-1/2}\int_0^1 e^{-(2-s)\sqrt{-A}}f(s)ds\Big\},\end{aligned}$$

$$\xi_1 = (I-Z)^{-1}\left\{-e^{-\sqrt{-A}}u_0 + \frac{1}{2}(-A)^{-1/2}\int_0^1 e^{-(1-s)\sqrt{-A}}f(s)ds \right.$$
$$\left. -\frac{1}{2}(-A)^{-1/2}\int_0^1 e^{-(1+s)\sqrt{-A}}f(s)ds\right\},$$

where $Z = e^{-2\sqrt{-A}}$. Hence

$$Au(x) = Ae^{-x\sqrt{-A}}\xi_0 + Ae^{-(1-x)\sqrt{-A}}\xi_1 + \frac{1}{2}L(x,f) + \frac{1}{2}L\left(1-x, f(1-\cdot)\right).$$

Now from (3.5) and Lemma 3.1 we deduce that $x \to Ae^{-x\sqrt{-A}}\xi_0 \in L^p(0,1;X)$ since

$$Ae^{-x\sqrt{-A}}\xi_0 = (I-Z)^{-1}M(x,u_0) - \frac{1}{2}(I-Z)^{-1}\mathcal{L}(x,f)$$
$$+\frac{1}{2}(I-Z)^{-1}e^{-\sqrt{-A}}\mathcal{L}(x,f(1-\cdot)).$$

Similarly $x \to Ae^{-(1-x)\sqrt{-A}}\xi_1 \in L^p(0,1;X)$. Then, due to (3.4), $Au \in L^p(0,1;X)$.

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4 General case

4.1 First approach: B generates a group

In this paragraph we assume (1.3)∼(1.9).

THEOREM 4.1 *Assume (1.3)∼(1.9). If*

$$f \in L^p(0,1;X), \quad 1 < p < +\infty, \text{ and } u_0, u_1 \in (D(A), X)_{1/(2p),p},$$

then Problem (1.1)–(1.2) has a unique strict solution u, that is

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), \quad u' \in L^p(0,1;D(B)),$$

and satisfies (1.1)–(1.2).

Proof. Replacing A by $A-B^2$ in Theorem 3.1, we see that Problem

$$\begin{cases} v''(x) + (A-B^2)v(x) = e^{xB}f(x), & x \in (0,1), \\ v(0) = u_0, \ v(1) = e^B u_1, \end{cases}$$

has a strict solution

$$v \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A-B^2)).$$

Then, we set

$$u(x) = e^{-xB}v(x), \quad x \in (0,1),$$

and it is easy to check that u has the desired properties.

4.2 Second approach

In this case, we assume (1.3)∼(1.7) together with (1.10)∼(1.13). Let us recall that $(T_0(x))_{x\geqslant 0}$ and $(T_1(x))_{x\geqslant 0}$ are the analytic semigroups generated by

$$-B-(B^2-A)^{1/2} \quad \text{and} \quad B-(B^2-A)^{1/2}.$$

Let $g\in L^p(0,1;X)$. Set

$$L_0(x,g)=\left(B+(B^2-A)^{1/2}\right)\int_0^x T_0(x-s)g(s)ds$$

$$L_1(x,g)=\left(B-(B^2-A)^{1/2}\right)\int_0^x T_1(x-s)g(s)ds,$$

and for $i,j\in\{0,1\}$

$$\begin{aligned}L_{i,j}(x,g)&=\left(B+(B^2-A)^{1/2}\right)\left(B^2-A\right)^{-1/2}\\&\quad\times\left(B-(B^2-A)^{1/2}\right)T_i(x)\int_0^1 T_j(s)g(s)ds.\end{aligned}$$

Applying again the Dore-Venni Theorem, as for (3.4), we get

$$L_0(\cdot,g),L_1(\cdot,g)\in L^p(0,1;X).$$

We have also, for any $i,j\in\{0,1\}$

$$L_{i,j}(\cdot,g)\in L^p(0,1;X),$$

since, for example

$$\begin{aligned}&L_{0,1}(x,g)\\&=\left(B-(B^2-A)^{1/2}\right)\left(B^2-A\right)^{-1/2}L_0(x,T_0(\cdot)T_1(\cdot)g(\cdot))\\&\quad+\left(B+(B^2-A)^{1/2}\right)\left(B^2-A\right)^{-1/2}T_0(x)T_1(x)L_1\left(1-x,g(1-\cdot)\right).\end{aligned}$$

Moreover, replacing $\sqrt{-A}$ by $\pm B-(B^2-A)^{1/2}$ in Lemma 3.1 we obtain

$$M_0(\cdot,w_0)=\left(B+(B^2-A)^{1/2}\right)^2T_0(\cdot)w_0\in L^p(0,1;X)$$

$$M_1(\cdot,w_1)=\left(B-(B^2-A)^{1/2}\right)^2T_1(\cdot)w_1\in L^p(0,1;X),$$

provided that

$$w_0\in\left(D\left(\left(B+(B^2-A)^{1/2}\right)^2\right),X\right)_{1/(2p),p},$$

$$w_1\in\left(D\left(\left(B-(B^2-A)^{1/2}\right)^2\right),X\right)_{1/(2p),p}.$$

Note that, by the well known Reiteration Theorem, we have, taking into account (1.5) and (1.7),

$$\left(D\left(\left(B \pm (B^2 - A)^{1/2}\right)^2\right), X\right)_{1/(2p),p} = \left(D(A), X\right)_{1/(2p),p}.$$

The main result in this paper is the following.

THEOREM 4.2 *Assume (1.3)∼(1.7) and (1.10)∼(1.13). If*

$$f \in L^p(0,1;X) \text{ with } 1 < p < +\infty, \text{ and } u_0, u_1 \in (D(A), X)_{1/(2p),p},$$

then Problem (1.1)–(1.2) has a unique strict solution u, that is

$$u \in W^{2,p}(0,1;X) \cap L^p(0,1;D(A)), \quad u' \in L^p(0,1;D(B)),$$

and satisfies (1.1)–(1.2).

Proof. We can suppose that $u_1 = 0$. Due to (1.15) and (2.2) we get, for $x \in (0,1)$,

$$\begin{aligned} AT_0(x)\xi_0 = {} & \left(B + (B^2 - A)^{1/2}\right)\left(B - (B^2 - A)^{1/2}\right)^{-1}(I - Z)^{-1}M_0(x, u_0) \\ & + \frac{1}{2}(I - Z)^{-1}L_{0,1}(x, f) - \frac{1}{2}(I - Z)^{-1}T_1(1)L_{0,0}\left(x, f(1 - \cdot)\right), \end{aligned}$$

hence $AT_0(\cdot)\xi_0 \in L^p(0,1;X)$. Similarly $AT_1(1 - \cdot)\xi_1 \in L^p(0,1;X)$. Finally, using (2.1), we can deduce that

$$\begin{aligned} Au(x) = {} & AT_0(x)\xi_0 + AT_1(1 - x)\xi_1 \\ & - \frac{1}{2}\left(B - (B^2 - A)^{1/2}\right)\left(B^2 - A\right)^{-1/2}L_0(x, f) \\ & - \frac{1}{2}\left(B + (B^2 - A)^{1/2}\right)\left(B^2 - A\right)^{-1/2}L_1\left(1 - x, f(1 - \cdot)\right), \end{aligned}$$

thus $Au \in L^p(0,1;X)$. To conclude we show that $Bu' \in L^p(0,1;X)$ by writing

$$\begin{aligned} Bu'(x) = {} & -B\left(B - (B^2 - A)^{1/2}\right)^{-1}AT_0(x)\xi_0 \\ & - B\left(B + (B^2 - A)^{1/2}\right)^{-1}AT_1(1 - x)\xi_1 \\ & + \frac{1}{2}B\left(B^2 - A\right)^{-1/2}L_0(x, f) + \frac{1}{2}B\left(B^2 - A\right)^{-1/2}L_1\left(1 - x, f(1 - .)\right). \end{aligned}$$

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5 Examples

Example 1. Let $X = L^q(\Omega)$, $1 < q < \infty$, where Ω is a domain in $\mathbb{R}^n$, $n \geqslant 1$, with C^2-boundary $\partial\Omega$. More precisely Ω is either $\mathbb{R}^n$, or the half space

$\mathbb{R}^n_+$, or a bounded domain with C^2-boundary, or an exterior domain with a C^2-boundary. Take as A the operator in X defined by

$$D(A) = W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), \quad A = \sum_{j,k=1}^{n} \frac{\partial}{\partial y_j}\left(a_{jk}\frac{\partial}{\partial y_k}\right) - \delta, \quad \delta \geqslant 0,$$

where $a = (a_{jk})$ and $b_k = -\sum_{j=1}^{n} \frac{\partial a_{jk}}{\partial y_j}$, $k = 1, 2, ..., n$, fulfill assumptions $(A1) \sim (A3)$ in the paper of Prüss-Sohr [17], i.e.,

(A1) $a(x) = (a_{jk}(x))$ is a real symmetric matrix for all $x \in \bar{\Omega}$ and there is $a_0 > 0$ such that $a_0 \leq a(x)\xi \cdot \xi \leq a_0^{-1}$ for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$;

(A2) $a_{jk} \in C^{\infty}(\bar{\Omega}))$ for some $\alpha \in (0,1)$ and, when Ω is unbounded, $a_{jk}^{\infty} = \lim_{|x|\to\infty} a_{jk}(x)$ exists and there is a constant $C > 0$ such that

$$|a_{jk}(x) - a_{jk}^{\infty}| \leq C|x|^{-\alpha} \text{ for all } x \in \Omega \text{ with } |x| \geq 1, \quad j,k = 1, \ldots, n;$$

(A3) $\dfrac{\partial a_{jk}}{\partial x_k} \in L^{r_k}(\Omega),\ p \leq r_k \leq \infty,\ r_k > n,\ j,k = 1, \ldots, n.$

In addition, it is supposed that either Ω is bounded or $\delta > 0$. Then $-A$ has bounded imaginary powers with estimates (3.3). Therefore Theorem 3.1 applies and we get

PROPOSITION 5.1 *Under the assumptions above, let* $p, q \in]1, \infty[$, $f \in L^p(0,1; L^q(\Omega))$ *and*

$$u_0, u_1 \in \left(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), L^q(\Omega)\right)_{1/(2p),p}.$$

Then Problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + \displaystyle\sum_{j,k=1}^{n} \frac{\partial}{\partial y_j}\left(a_{jk}\frac{\partial u}{\partial y_k}\right)(x,y) - \delta u(x,y) \\ \qquad = f(x,y), \qquad (x,y) \in (0,1)\times\Omega, \\ u(0,y) = u_0(y),\ u(1,y) = u_1(y), \quad y \in \Omega, \\ u(x,\sigma) = 0, \qquad (x,\sigma) \in (0,1)\times\partial\Omega, \end{cases} \tag{5.1}$$

has a unique strict solution u, *that is*

$$u \in W^{2,p}(0,1; L^q(\Omega)) \cap L^p(0,1; W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)),$$

and satisfies (5.1).

Note that here the interpolation space

$$\left(W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega), L^q(\Omega)\right)_{1/(2p),p}$$

is, for bounded domain, the Besov space

$$\left\{u \in \mathcal{B}_{q,p}^{2-1/p}(\Omega) : u_{|\partial\Omega} = 0\right\},$$

see Triebel [20], p. 321 (for the case $\Omega = \mathbb{R}^n$ or $\mathbb{R}_+^n$ one applies Triebel, again, Theorem 5.3.3. p. 373).

Example 2. Take $X = L^p(\mathbb{R}), 1 < p < +\infty$. Define the operators A, B by

$$\begin{cases} D(A) = W^{2,p}(\mathbb{R}), & Au = au'' - cu, \\ D(B) = W^{1,p}(\mathbb{R}), & Bu = bu', \end{cases}$$

where $a - b^2 > 0$ and $c > 0$. Then

$$D(B^2 - A) = W^{2,p}(\mathbb{R}) \text{ and } D((B^2 - A)^{1/2}) = W^{1,p}(\mathbb{R}),$$

so (1.5)∼(1.7) are verified. Moreover a simple computation shows that (1.4) holds. Assumption (1.8) is also satisfied. At last, in virtue of Theorem C, p. 167, in Prüss-Sohr [17] again one sees that $B^2 - A$ has bounded imaginary powers with estimates (1.9).

Applying Theorem 4.1 of the first approach, we get

PROPOSITION 5.2 *Let* $p \in]1, +\infty[$, $f \in L^p(0,1;X)$ *and* $u_0, u_1 \in \left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{1/(2p),p}$. *Then Problem*

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + 2b\dfrac{\partial^2 u}{\partial x \partial y}(x,y) + a\dfrac{\partial^2 u}{\partial y^2}(x,y) - cu(x,y) \\ \qquad\qquad = f(x,y), \quad (x,y) \in]0,1[\times \mathbb{R}, \\ u(0,y) = u_0(y), \quad u(1,y) = u_1(y), \quad y \in \mathbb{R}, \end{cases} \tag{5.2}$$

has a unique strict solution u, *that is*

$$u \in W^{2,p}(0,1;L^p(\mathbb{R})) \cap L^p(0,1;W^{2,p}(\mathbb{R})), \quad u' \in L^p(0,1;W^{1,p}(\mathbb{R})),$$

and satisfies (5.2).

Note that here the interpolation space $\left(W^{2,p}(\mathbb{R}), L^p(\mathbb{R})\right)_{1/(2p),p}$ coincides with the following Besov space $\mathcal{B}_p^{2-1/p}(\mathbb{R})$, see Grisvard [10], Teorema 7, p. 681. Of course one could establish a more general result in the space

$$L^p(0,1;L^q(\mathbb{R})), \quad 1 < p, q < +\infty,$$

(cf. **Example 1**).

Example 3. Let Ω be a bounded domain in $\mathbb{R}^n, n \geqslant 1$, with a smooth boundary $\partial\Omega$ and $X = L^p(\Omega)$, $1 < p < +\infty$. Define the operators A, B by

$$D(A) = \left\{u \in W^4(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\right\}, \quad Au = b\Delta^2 u,$$

where $b < 0$ and

$$D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad Bu = \Delta u.$$

Then B generates a bounded analytic semigroup and $0 \in \rho(B)$. We deduce that $\left(B^2\right)^{1/2} = -B$, and

$$\pm B - (B^2 - A)^{1/2} = \pm B + (1-b)^{1/2} B = \left((1-b)^{1/2} \pm 1\right) B,$$

which generates an analytic semigroup. On the other hand, the results from Seeley [18] guarantee that (1.13) holds. Then Theorem 4.2 runs and we can handle the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + 2\Delta \dfrac{\partial u}{\partial x}(x,y) + b\Delta^2 u(x,y) = f(x,y), \qquad (x,y) \in (0,1)\times\Omega, \\ u(0,y) = u_0(y), \quad y \in \Omega, \\ u(1,y) = u_1(y), \quad y \in \Omega, \\ u(x,\xi) = \Delta_y u(x,\xi) = 0, \qquad (x,\xi) \in (0,1)\times\partial\Omega, \end{cases}$$

provided that $f \in L^p(0,1; L^p(\Omega))$ and $u_0, u_1 \in (D(A), L^p(\Omega))_{1/(2p),p}$.

We need to describe this interpolation space. To this end, we recall from [14], Théorème 3.2, p.59, that if Λ generates a bounded C_0-semigroup in the Banach space X, $m \in \mathbb{N}$, $\theta \in (0,1)$, $1 < p < \infty$, $(1-\theta)m = j + \eta$, where j is an integer ≥ 0 and $0 < \eta < 1$, then

$$(D(\Lambda^m), X)_{\theta,p} = \{a \in D(\Lambda^j);\ \Lambda^j a \in (X, D(\Lambda))_{(1-\theta)m-j,p}\}.$$

In our case, $\Lambda = B$, $m = 2$, $\theta = 1/(2p)$, so that $j = 1$, $\eta = 1 - 1/p$. Therefore,

$$(D(A), L^p(\Omega))_{1/(2p),p} = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega);\ \Delta u \in (L^p(\Omega), D(B))_{1-1/p,p}\}.$$

On the other hand, by using [20], p. 321, we have

$$(L^p(\Omega), D(B))_{1-1/p,p} = \mathcal{B}_{p,p}^{2(1-1/p)}(\Omega), \text{ if } 1 < p < 3/2,$$

and

$$(L^p(\Omega), D(B))_{1-1/p,p} = \{v \in \mathcal{B}_{p,p}^{2(1-1/p)}(\Omega);\ v_{|\partial\Omega} = 0\}, \text{ if } p > 3/2.$$

If $p = 3/2$, then (by [20], p. 319-321)

$$(L^{3/2}(\Omega), D(B))_{1/3,3/2} = \{v \in \mathcal{B}_{3/2,3/2}^{2/3}(\Omega);\ \int_\Omega d(x)^{-1} |v(x)|^{3/2} dx < \infty\},$$

where $d(x)$ denotes the distance of the point $x \in \Omega$ to the boundary and $\mathcal{B}_{p,q}^s(\Omega)$ are the Besov spaces, $s > 0$, $p, q \geq 1$. Thus we have fully characterized $(D(A), L^p(\Omega))_{1/(2p),p}$.

Example 4. (Periodic boundary conditions).
Take $X = L^2(0,1)$ and consider the operator T in X defined by

$$D(T) = \left\{ f \in H^1(0,1) : f(0) = f(1) \right\}, \quad Tf = if'.$$

It is well known that T is self-adjoint and its spectrum is $\sigma(T) = 2\pi\mathbb{Z}$ (see Miklavčič [16], p. 75). Then

$$D(T^2) = \left\{ f \in H^2(0,1) : f(0) = f(1),\ f'(0) = f'(1) \right\}, \quad T^2 f = -f'',$$

and T^2 is positive, self-adjoint. We take $B = -iT$ (generating a strongly continuous group) and introduce A by

$$D(A) = D(T^2), \quad Af = (-2T^2 - aI)f = 2f'' - af$$

(with $a > 0$).

Then $B^2 - A = T^2 + aI$, with domain $D(T^2)$, is a positive self-adjoint operator. Thus $D(T)$ coincides with the complex interpolation space $\left[X, D(T^2)\right]_{1/2}$, (see Triebel [20], p. 143) and $(T^2 + aI)^{1/2}$ is positive self-adjoint.

Hence Theorem 4.1 enables us to solve the boundary value Problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) + 2\dfrac{\partial^2 u}{\partial y \partial x}(x,y) + 2\dfrac{\partial^2 u}{\partial y^2}(x,y) - au(x,y) \\ = f(x,y), \qquad (x,y) \in (0,1)\times(0,1), \\ u(0,y) = u_0(y),\ u(1,y) = u_1(y), \qquad 0 < y < 1, \\ u(x,0) = u(x,1), \dfrac{\partial u}{\partial y}(x,0) = \dfrac{\partial u}{\partial y}(x,1), \qquad 0 < x < 1, \end{cases}$$

provided that $f \in L^p(0,1; L^2(0,1))$ and $u_0, u_1 \in \left(D(A), L^2(0,1)\right)_{1/(2p),p}$. We can then apply the preceding argument, reducing such a space to an interpolation space between $D(T)$ and $L^2(0,1)$.

Example 5. Let H be a Hilbert space and B a strictly positive, self-adjoint operator in X. Take $A = -B^3$. Then $B^2 - A$ is strictly positive self-adjoint, and

$$D\big((B^2 - A)^{1/2}\big) = D\big(B^{3/2}\big).$$

Moreover $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup in X and we have also $D(A) \subsetneq D(B^2)$ (for details see **Example 3** in [7]). Since $\pm B + (B^2 - A)^{1/2}$ is a positive operator, Theorem 4.2 works.

As an example, we take A, B defined in $X = L^2(\Omega)$ by

$$D(B) = H_0^1(\Omega) \cap H^2(\Omega), \quad B = -\Delta,$$

$$D(A) = \left\{ u \in H^6(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = \Delta^2 u_{|\partial\Omega} = 0 \right\}, \quad A = \Delta^3,$$

where Ω is a bounded domain in $\mathbb{R}^q, q \geq 1$, with a C^2-boundary $\partial\Omega$ and we can then handle the boundary problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}(x,y) - 2\Delta\dfrac{\partial u}{\partial x}(x,y) + \Delta^3 u(x,y) = f(x,y), \qquad (x,y) \in (0,1)\times\Omega, \\ u(0,y) = u_0(y),\ u(1,y) = u_1(y), \quad y \in \Omega, \\ u(x,\sigma) = \Delta u(x,\sigma) = \Delta^2 u(x,\sigma) = 0, \qquad (x,\sigma) \in (0,1)\times\partial\Omega, \end{cases}$$

provided that $f \in L^2((0,1)\times\Omega)$ and $u_0, u_1 \in (D(A), L^2(\Omega))_{1/4,2}$. This interpolation space can be characterized using [20], Theorem 4.4.1 (6), p.321 and [14], p.59, again. Precisely, $(D(A), L^2(\Omega))_{1/4,2}$ coincides with

$$\begin{aligned} &\left\{u \in H^4(\Omega):\ u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0,\ \Delta^2 u \in (L^2(\Omega), H_0^1(\Omega)\cap H^2(\Omega))_{1/4,2}\right\} \\ = &\Big\{u \in H^4(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0,\ \Delta^2 u \in B_{2,2}^{1/2}(\Omega), \\ &\qquad \int_\Omega d^{-1}(x)|\Delta^2 u(x)|^2 dx < \infty\Big\}, \end{aligned}$$

where $B_{2,2}^{1/2}(\Omega)$ is a Besov space and $d(x)$ denotes the distance of $x \in \Omega$ from the boundary $\partial\Omega$.

Here we have taken advantage from self-adjointness property of the involved operators guaranteeing the boundedness of imaginary powers. More sophisticated examples can be described using the perturbation results by Dore-Venni [5] and Prüss-Sohr [17].

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Degenerate integrodifferential equations of parabolic type

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Abstract We consider the initial value problem for a possibly degenerate integrodifferential equation in $L^2(\Omega)$

$$D_t(M(t)u(t)) + L(t)u(t) + \int_0^t B(t,s)u(s)ds = f(t), \quad 0 < t \leq T,$$
$$M(0)u(0) = Mu_0,$$

where $M(t) = M_0 M_1(t)$ is the multiplication operator by the function $m(x,t) = m_0(x)m_1(x,t)$, $m_0(x) \geq 0, m_1(x,t) \geq c > 0$, $L(t)$ is the realization in $L^2(\Omega)$ of a second-order strongly elliptic operator in divergence form with Dirichlet or Neumann boundary conditions for all t, and $B(t,s)$ is a linear differential operator of order ≤ 2 for each (t,s), $0 \leq s \leq t \leq T$, Ω being a bounded open set in $\mathbf{R}^n$ with a smooth boundary.
We also establish a corresponding result in $L^p(\Omega), 1 < p < 3/2$, related to Dirichlet boundary condition, only.

1 Introduction

This paper is concerned with the initial value problem of the following degenerate integrodifferential equation

$$\begin{aligned} &D_t(M(t)u(t)) + L(t)u(t) + \int_0^t B(t,s)u(s)ds = f(t), \\ &M(0)u(0) = M(0)u_0. \end{aligned} \tag{1.1}$$

Here, $M(t) = M_0 M_1(t)$ is the multiplication operator by the function $m(x,t) = m_0(x)m_1(x,t)$:

$$(M(t)u)(x) = m(x,t)u(x),$$
$$(M_0 u)(x) = m_0(x)u(x),$$
$$(M_1(t)u)(x) = m_1(x,t)u(x),$$

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$L(t)$ is the realization of a second order strongly elliptic operator in divergence form

$$\mathcal{L}(t) = -\sum_{i,j=1}^{n} D_{x_i}(a_{i,j}(x,t)D_{x_j}) + a_0(x,t), \quad t \in [0,T], \quad D_{x_i} = \frac{\partial}{\partial x_i},$$

in $L^2(\Omega)$ with the Dirichlet or Neumann boundary condition for each $t \in [0,T]$ and $B(t,s)$ is linear differential operator of order not exceeding two for each (t,s) such that $0 \leq s \leq t \leq T$, where Ω is a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The sesquilinear form associated with $\mathcal{L}(t)$ is denoted by $a(t;u,v)$.

Assumptions

(I) $\mu_1|\xi|^2 \leq \sum_{i,j=1}^{n} a_{i,j}(x,t)\xi_i\xi_j \leq \mu_2|\xi|^2$, for all $(x,t,\xi) \in \Omega \times [0,T] \times \mathbf{R}^n$ for some positive constants μ_1 and μ_2, $\mu_1 < \mu_2$;

(II) $a_0(x,t) \geq \gamma > 0$ for all $(x,t) \in \Omega \times [0,T]$;

(III) $m_0 \in L^\infty(\Omega)$, $m_0(x) \geq 0$ a.e. in Ω;

(IV) $m_1 \in C^{1,\rho}([0,T]; W^{1,\infty}(\Omega))$ and $\inf_{x\in\Omega, t\in[0,T]} m_1(x,t) > 0$;

(V) for each $u, v \in H^1(\Omega)$, $a(t;u,v)$ is differentiable in t, and

$$|a(t;u,v)| \leq C\|u\|_{H^1}\|v\|_{H^1},$$

$$|\dot{a}(t;u,v) - \dot{a}(s;u,v)| \leq C|t-s|^\rho \|u\|_{H^1}\|v\|_{H^1}, \quad t,s \in (0,T),$$

where $\dot{a} = D_t a$;

(VI) $1/2 < \rho \leq 1$;

(VII) the coefficients of $B(t,s)$ are continuous in $\overline{\Omega} \times [0,T]$ and uniformly Hölder continuous of order ρ.

It is also assumed that the coefficients of $\mathcal{L}(t)$ are sufficiently smooth, and $0 \in \rho(L(t))$ for all $t \in [0,T]$.
We stress that, according to assumptions (I) and (II), the zeros of m are *time independent*.
Before solving (1.1) the problem without the integral term

$$\begin{aligned} &D_t(M(t)u(t)) + L(t)u(t) = f(t), \\ &M(0)u(0) = M(0)u_0 \end{aligned} \tag{1.2}$$

is considered. By introducing the new unknown variable $v(t) = M(t)u(t)$ this problem is transformed to

$$\begin{aligned} &v'(t) + A(t)v(t) \ni f(t), \\ &v(0) = v_0, \end{aligned} \tag{1.3}$$

where $A(t) = L(t)M(t)^{-1}$ is a possibly multivalued operator and $v_0 = M(0)u_0$. According to the result of [2] $-A(t)$ generates an infinitely many times differentiable semigroup for each $t \in [0, T]$ in $L^2(\Omega)$.

In order to construct the fundamental solution $U(t, s)$ in $L^2(\Omega)$ to the problem (1.3) it is convenient to consider the same problem in the space of negative norm $H^{-1}(\Omega) = H_0^1(\Omega)^*$ or $H^1(\Omega)^*$ according as the boundary condition is of Dirichlet type or Neumann type. Let $\widetilde{L}(t)$ be the operator defined by

$$(\widetilde{L}(t)u, v)_{H^{-1} \times H_0^1} = a(t; u, v), \quad u, v \in H_0^1(\Omega)$$

in case of the Dirichlet condition, and

$$(\widetilde{L}(t)u, v)_{(H^1)^* \times H^1} = a(t; u, v), \quad u, v \in H^1(\Omega)$$

in case of the Neumann condition. Let $\widetilde{A}(t) = \widetilde{L}(t)M(t)^{-1}$. It is shown in [3] that $\widetilde{A}(t)$ satisfies the parabolicity condition with $\alpha = \beta = 1$. Making use of the fact that

$$D(\widetilde{L}(t)) = \begin{cases} H_0^1(\Omega) & \text{in case of the Dirichlet condition} \\ H^1(\Omega) & \text{in case of the Neumann condition} \end{cases}$$

is independent of t the fundamental solution $\widetilde{U}(t, s)$ to the problem

$$\begin{aligned} &v'(t) + \widetilde{A}(t)v(t) \ni f(t), \\ &v(0) = v_0 \end{aligned} \tag{1.4}$$

can be constructed applying the method of Kato and Tanabe [4]. The desired fundamental solution is obtained by $U(t, s) = \widetilde{U}(t, s)|_{L^2(\Omega)}$.

The norms of $H^{-1}(\Omega)$, $H^1(\Omega)^*$, $\mathcal{L}(H^{-1}(\Omega))$ and $\mathcal{L}(H^1(\Omega)^*)$ are all simply denoted by $\| \cdot \|_*$.

---

2 Equations without integral term

It was shown in [3] and [2] that the following inequalities hold:

$$\|(\lambda + \widetilde{A}(t))^{-1}\|_* = \|M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\|_* \le C_0(1 + |\lambda|)^{-1}, \tag{2.1}$$

$$\begin{aligned} &\|(\lambda + \widetilde{A}(t))^{-1} f\|_{L^2} \\ &= \|M(t)(\lambda M(t) + \widetilde{L}(t))^{-1} f\|_{L^2} \le C_0(1 + |\lambda|)^{-1/2}\|f\|_* \end{aligned} \tag{2.2}$$

for $\lambda \in \Sigma = \{\lambda \in \mathbb{C}; \mathrm{Re}\lambda \ge -c_0(|\mathrm{Im}\lambda| + 1)\}$, $t \in [0, T]$ and some positive constants C_0 and c_0. Hence $-\widetilde{A}(t)$ generates in $H^{-1}(\Omega)$ or in $H^1(\Omega)^*$ an analytic semigroup $\exp(-\tau\widetilde{A}(t))$ satisfying

$$\begin{aligned}&\|\exp(-\tau\widetilde{A}(t))\|_* \le C, \quad \|\exp(-\tau\widetilde{A}(t))f\|_{L^2} \le C\tau^{-1/2}\|f\|_*,\\ &\big\|D_\tau \exp(-\tau\widetilde{A}(t))f\big\|_{L^2} \le C\tau^{-3/2}\|f\|_*.\end{aligned} \tag{2.3}$$

$D(\widetilde{A}(t))$ is not dense unless $m_0(x) > 0$ a.e. Therefore

$$\lim_{\tau\downarrow 0}\|\exp(-\tau\widetilde{A}(t))v - v\|_* = 0$$

holds only for $v \in \overline{D(\widetilde{A}(t))} \equiv \overline{D(\widetilde{A}(0))}$.

LEMMA 2.1 *For* $\lambda \in \Sigma$, $t \in [0,T]$

$$\big\|D_t(\lambda + \widetilde{A}(t))^{-1}\big\|_* \le \frac{C}{|\lambda|}, \tag{2.4}$$

$$\big\|D_t(\lambda + \widetilde{A}(t))^{-1}f\big\|_{L^2} \le \frac{C}{|\lambda|^{1/2}}\|f\|_*. \tag{2.5}$$

Proof. As is easily seen

$$\begin{aligned}&D_t(\lambda + \widetilde{A}(t))^{-1} = D_t\big\{M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\big\}\\ &= \dot{M}(t)(\lambda M(t) + \widetilde{L}(t))^{-1}\\ &\quad - M(t)(\lambda M(t) + \widetilde{L}(t))^{-1}(\lambda\dot{M}(t) + \dot{\widetilde{L}}(t))(\lambda M(t) + \widetilde{L}(t))^{-1},\end{aligned} \tag{2.6}$$

where $\dot{M}(t) = D_tM(t)$ and $\dot{\widetilde{L}}(t) = D_t\widetilde{L}(t)$.
Note that $\dot{M}(t)$ is the multiplication operator by the function

$$D_tm(t) = m_0D_tm_1(t) = \frac{D_tm_1(t)}{m_1(t)}m_0m_1(t) = \frac{D_tm_1(t)}{m_1(t)}m(t),$$

and

$$\dot{\widetilde{L}}(t) = \dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}\widetilde{L}(t).$$

The multiplications by $D_tm_1(x,t)/m_1(x,t)$ and $\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}$ define uniformly bounded operators from $H^{-1}(\Omega)$ or $H^1(\Omega)^*$ to itself. Hence the result is established by using (2.1) and (2.2).

LEMMA 2.2 *For* $0 \le \tau < t \le T$ *and* $\lambda \in \Sigma$

$$\big\|D_t(\lambda + \widetilde{A}(t))^{-1} - D_\tau(\lambda + \widetilde{A}(\tau))^{-1}\big\|_* \le C\frac{(t-\tau)^\rho}{|\lambda|}.$$

Proof. The assertion can be proved by the argument of the proof of the previous lemma, and the Hölder continuity of the function $D_t m_1(t)/m_1(t)$ in $W^{1,\infty}(\Omega)$ and of the operator valued function $\dot{\widetilde{L}}(t)\widetilde{L}(t)^{-1}$.

With the aid of Lemmas 2.1 and 2.2 one can construct the fundamental solution to the problem

$$\begin{aligned} &v'(t) + \widetilde{A}(t)v(t) \ni f(t), \quad 0 < t \leq T, \\ &v(0) = v_0, \end{aligned} \tag{2.7}$$

by the method of [4]:

$$\begin{aligned} \widetilde{U}(t,s) &= \exp(-(t-s)\widetilde{A}(t)) + \int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}(\tau,s)d\tau, \\ \widetilde{\Phi}(t,s) &= \widetilde{\Phi}_1(t,s) + \int_s^t \widetilde{\Phi}_1(t,\tau)\widetilde{\Phi}(\tau,s)d\tau, \\ \widetilde{\Phi}_1(t,s) &= -(D_t + D_s)\exp(-(t-s)\widetilde{A}(t)) \\ &= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}D_t(\lambda + \widetilde{A}(t))^{-1}d\lambda. \end{aligned}$$

LEMMA 2.3 *For* $0 \leq s < \tau < t \leq T$

$$\|\widetilde{\Phi}_1(t,s)\|_* \leq C, \quad \|\widetilde{\Phi}(t,s)\|_* \leq C, \tag{2.8}$$

$$\|\widetilde{\Phi}_1(t,s) - \widetilde{\Phi}_1(\tau,s)\|_* \leq C\Big\{(t-\tau)^\rho + \log\frac{t-s}{\tau-s}\Big\}, \tag{2.9}$$

$$\|\widetilde{\Phi}_1(t,s)f\|_{L^2} \leq C(t-s)^{-1/2}\|f\|_*. \tag{2.10}$$

Proof. The above inequalities are simple consequences of Lemmas 2.1 and 2.2.

Let $0 < \varepsilon < t - s$. Then, with the aid of the usual argument

$$\begin{aligned} &D_t\int_s^{t-\varepsilon}\exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,s)f d\tau \\ &= \exp(-\varepsilon\widetilde{A}(t))\widetilde{\Phi}_1(t-\varepsilon,s)f - \int_s^{t-\varepsilon}\widetilde{\Phi}_1(t,\tau)\widetilde{\Phi}_1(\tau,s)f d\tau \\ &\quad - \int_s^{t-\varepsilon}\frac{\partial}{\partial\tau}\exp(-(t-\tau)\widetilde{A}(t))(\widetilde{\Phi}_1(\tau,s) - \widetilde{\Phi}_1(t,s))f d\tau \\ &\quad - \exp(-\varepsilon\widetilde{A}(t))\widetilde{\Phi}_1(t,s)f + \exp(-(t-s)\widetilde{A}(t))\widetilde{\Phi}_1(t,s)f. \end{aligned}$$

In view of the inequalities (2.3) and (2.9)

$$\begin{aligned}
&\|\exp(-\varepsilon\widetilde{A}(t))\widetilde{\Phi}_1(t-\varepsilon,s)f-\exp(-\varepsilon\widetilde{A}(t))\widetilde{\Phi}_1(t,s)f\|_{L^2}\\
&=\|\exp(-\varepsilon\widetilde{A}(t))(\widetilde{\Phi}_1(t-\varepsilon,s)-\widetilde{\Phi}_1(t,s))f\|_{L^2}\\
&\le C\varepsilon^{-1/2}\|(\widetilde{\Phi}_1(t-\varepsilon,s)-\widetilde{\Phi}_1(t,s))f\|_*\\
&\le C\varepsilon^{-1/2}\Big(\varepsilon^\rho+\log\frac{t-s}{t-\varepsilon-s}\Big)\|f\|_*\to 0
\end{aligned}$$

as $\varepsilon\to 0$. Hence one obtains

$$\begin{aligned}
D_t\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,s)f\,d\tau &= -\int_s^t \widetilde{\Phi}_1(t,\tau)\widetilde{\Phi}_1(\tau,s)f\,d\tau\\
&-\int_s^t D_\tau \exp(-(t-\tau)\widetilde{A}(t))(\widetilde{\Phi}_1(\tau,s)-\widetilde{\Phi}_1(t,s))f\,d\tau\\
&+\exp(-(t-s)\widetilde{A}(t))\widetilde{\Phi}_1(t,s)f,
\end{aligned}$$

and

$$\Big\|D_t\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,s)f\,d\tau\Big\|_{L^2}\le C(t-s)^{-1/2}\|f\|_*. \tag{2.11}$$

In view of (2.3) and (2.8) it is evident that

$$\Big\|\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,s)f\,d\tau\Big\|_{L^2}\le C(t-s)^{1/2}\|f\|_*. \tag{2.12}$$

By virtue of (2.11) and (2.12) one gets

$$\begin{aligned}
&D_t\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}(\tau,s)f\,d\tau\\
&=D_t\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,s)f\,d\tau\\
&+\int_s^t D_t\int_\sigma^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}_1(\tau,\sigma)\,d\tau\,\widetilde{\Phi}(\sigma,s)f\,d\sigma,
\end{aligned}$$

and

$$\Big\|D_t\int_s^t \exp(-(t-\tau)\widetilde{A}(t))\widetilde{\Phi}(\tau,s)f\,d\tau\Big\|_{L^2}\le C(t-s)^{-1/2}\|f\|_*. \tag{2.13}$$

With the aid of (2.3), (2.8), (2.10) and (2.13) it is easily shown that

$$\begin{aligned}
&\|\widetilde{U}(t,s)f\|_{L^2}\le C(t-s)^{-1/2}\|f\|_*,\\
&\|D_t\widetilde{U}(t,s)f\|_{L^2}\le C(t-s)^{-3/2}\|f\|_*.
\end{aligned}$$

By using the identity

$$D_\tau \exp(-\tau \widetilde{A}(t))\widetilde{A}(t)^{-1} = -\exp(-\tau \widetilde{A}(t))$$

it is not difficult to show

$$\widetilde{A}(t)^{-1} D_t \widetilde{U}(t,s)f = -\widetilde{U}(t,s)f.$$

This implies

$$D_t \widetilde{U}(t,s)f + \widetilde{A}(t)\widetilde{U}(t,s)f \ni 0.$$

Hence the operator valued function $U(t,s)$ defined by

$$U(t,s) = \widetilde{U}(t,s)|_{L^2(\Omega)}$$

satisfies

$$D_t U(t,s)f + A(t)U(t,s)f \ni 0, \quad 0 \le s < t \le T,$$

for any $f \in L^2(\Omega)$. It is easy to show that for

$$v_0 \in D(\widetilde{A}(s)) = \{m(s)u,\, u \in H_0^1(\Omega) \text{ or } H^1(\Omega)\}(= D(\widetilde{A}(0)))$$

one has

$$\|\exp(-(t-s)A(s))v_0 - v_0\|_{L^2} \le C(t-s)^{1/2}\|v_0\|_{D(\widetilde{A}(s))},$$

$$\|\{\exp(-(t-s)A(t)) - \exp(-(t-s)A(s))\}v_0\|_{L^2} \le C(t-s)^{1/2}\|v_0\|_*.$$

Hence it follows that

$$U(t,s)v_0 \to v_0 \text{ in } L^2(\Omega)$$

as $t \to s$ for $v_0 \in D(\widetilde{A}(s))$.

Let $f \in C^\rho([0,T]; H^{-1}(\Omega))$ or $f \in C^\rho([0,T]; H^1(\Omega)^*)$. Arguing as in the proof of (2.11) and using (2.3) and (2.8) one obtains that

$$D_t \int_0^t \exp(-(t-s)\widetilde{A}(t))f(s)ds = -\int_0^t \widetilde{\Phi}_1(t,s)f(s)ds$$
$$-\int_0^t D_s \exp(-(t-s)\widetilde{A}(t)) \cdot (f(s) - f(t))ds + \exp(-t\widetilde{A}(t))f(t),$$

and

$$\left\| D_t \int_0^t \exp(-(t-s)\widetilde{A}(t))f(s)ds \right\|_{L^2(\Omega)}$$
$$\le Ct^{1/2} \sup_{0 \le s \le t} \|f(s)\|_* + Ct^{\rho - 1/2} \sup_{0 \le s < t} \frac{\|f(t) - f(s)\|_*}{(t-s)^\rho} + Ct^{-1/2}\|f(t)\|_*.$$

From this and (2.13) it follows that $\int_0^t \widetilde{U}(t,s)f(s)ds$ is differentiable in t. Thus the following theorem is established.

THEOREM 2.1 *Let* $f \in C^\rho([0,T]; L^2(\Omega))$ *and* $v_0 \in D(\widetilde{A}(0))$*. Then*

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)ds$$

is a solution of the initial value problem

$$v'(t) + A(t)v(t) \ni f(t), \quad 0 < t \le T,$$
$$v(0) = v_0.$$

In order to show the uniqueness of the solution to the problem

$$\begin{aligned} &v'(t) + \widetilde{A}(t)v(t) \ni f(t), \quad 0 < t \le T, \\ &v(0) = v_0 \end{aligned} \tag{2.7}$$

we construct the operator valued function $\widetilde{V}(t,s)$ as follows:

$$\widetilde{V}(t,s) = \exp(-(t-s)\widetilde{A}(s)) + \int_s^t \widetilde{\Psi}(t,\tau)\exp(-(\tau-s)\widetilde{A}(s))d\tau,$$

$$\widetilde{\Psi}(t,s) = \widetilde{\Psi}_1(t,s) + \int_s^t \widetilde{\Psi}(t,\tau)\widetilde{\Psi}_1(\tau,s)d\tau,$$

$$\widetilde{\Psi}_1(t,s) = (D_t + D_s)\exp(-(t-s)\widetilde{A}(s)).$$

Just as in Lemma 2.3 it is seen that

$$\|\widetilde{\Psi}_1(t,s)\|_* \le C, \quad \|\widetilde{\Psi}(t,s)\|_* \le C.$$

It is not difficult to show that

$$D_s\widetilde{V}(t,s) \cdot \widetilde{A}(s)^{-1} = \widetilde{V}(t,s).$$

Let v be a solution of (2.7). Then

$$\begin{aligned} D_s(\widetilde{V}(t,s)v(s)) &= D_s\widetilde{V}(t,s) \cdot v(s) + \widetilde{V}(t,s)v'(s) \\ &= D_s\widetilde{V}(t,s) \cdot \widetilde{A}(s)^{-1}(f(s) - v'(s)) + \widetilde{V}(t,s)v'(s) \\ &= \widetilde{V}(t,s)(f(s) - v'(s)) + \widetilde{V}(t,s)v'(s) = \widetilde{V}(t,s)f(s). \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} &\exp(-(t-s)\widetilde{A}(s))v(s) - v(t) \\ &\quad = \{\exp(-(t-s)\widetilde{A}(s)) - \exp(-(t-s)\widetilde{A}(t))\}v(s) \\ &\qquad + \exp(-(t-s)\widetilde{A}(t))(v(s) - v(t)) \\ &\qquad + \exp(-(t-s)\widetilde{A}(t))v(t) - v(t) \to 0 \end{aligned}$$

as $s \uparrow t$, one has

$$\widetilde{V}(t,s)v(s) \to v(t)$$

as $s \uparrow t$. Hence integrating (2.14) from 0 to t one concludes

$$v(t) - \widetilde{V}(t,0)v_0 = \int_0^t \widetilde{V}(t,s)f(s)ds.$$

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3 Equations with integral term

Let $\Delta = \{(t,s); 0 \leq s < t \leq T\}$. Assume that the coefficients of $B(t,s)$ belong to $C^\rho(\overline{\Omega} \times \overline{\Delta})$ and are uniformly Hölder continuous functions of (t,s) in $\overline{\Delta}$. Let $K(t,s) = B(t,s)L(s)^{-1} \in \mathcal{L}(L^2(\Omega))$. By assumption $K(\cdot,\cdot) \in C^\rho(\overline{\Delta}; \mathcal{L}(L^2(\Omega)))$. The problem

$$D_t(M(t)u(t)) + L(t)u(t) + \int_0^t B(t,s)u(s)ds = f(t), \tag{3.1}$$

$$M(0)u(0) = M(0)u_0 \tag{3.2}$$

is rewritten as

$$\begin{aligned} &D_t(M(t)u(t)) + L(t)u(t) + \int_0^t K(t,s)L(s)u(s)ds = f(t), \\ &M(0)u(0) = M(0)u_0. \end{aligned} \tag{3.3}$$

Let $v(t) = M(t)u(t)$ be the new unknown function. Then $L(t)u(t) \in A(t)v(t)$, $L(s)u(s) \in A(s)v(s)$. In order to avoid that the multivalued operator appears in two places, one uses the idea due to Crandall and Nohel [1]. The convolution $F * G$ of two operator valued functions F and G and that $F * f$ of an operator valued function F and a vector valued function f are defined by

$$(F * G)(t,s) = \textstyle\int_s^t F(t,r)G(r,s)dr,$$

$$(F * f)(t) = \textstyle\int_0^t F(t,s)f(s)ds,$$

respectively. Then (3.3) is briefly rewritten as

$$(Mu)' + Lu + K * Lu = f. \tag{3.4}$$

Let $R : \overline{\Delta} \to \mathcal{L}(L^2(\Omega))$ be the solution of the integral equation

$$R + K + K * R = 0. \tag{3.5}$$

It is easily seen that $R \in C^\rho(\overline{\Delta}; \mathcal{L}(L^2(\Omega)))$ and $K * R = R * K$. Hence

$$R + K + R * K = 0. \tag{3.6}$$

Convoluting R with both sides of (3.4), one gets

$$R * (Mu)' + R * Lu + R * K * Lu = f. \tag{3.7}$$

Adding (3.4) and (3.7), and using (3.6) one obtains

$$(Mu)' + Lu = f + R * f - R * (Mu)'. \tag{3.8}$$

Let $v = Mu$ be the new unknown function. Then, in view of (3.8) one has

$$\begin{aligned} &v' + Av \ni f + R * f - R * v' \\ &v(0) = v_0 = M(0)u_0. \end{aligned} \tag{3.9}$$

This problem is transformed into the integrodifferential equation

$$\begin{aligned} v(t) &= U(t,0)v_0 + \int_0^t U(t,s)\{f(s) + (R * f)(s) - (R * v')(s)\}ds \\ &= g(t) - U * R * v'(t), \end{aligned} \tag{3.10}$$

where

$$g(t) = U(t,0)v_0 + \int_0^t U(t,s)\{f(s) + (R * f)(s)\}ds. \tag{3.11}$$

Differentiation of (3.10) yields

$$v'(t) = g'(t) - \int_0^t Q(t,s)v'(s)ds, \tag{3.12}$$

where

$$Q(t,s) = D_t(U * R)(t,s) = D_t \int_s^t U(t,r)R(r,s)dr.$$

The equation (3.12) is considered to be the integral equation to be satisfied by v'. Let w be the solution to the equation

$$w(t) = g'(t) - \int_0^t Q(t,s)w(s)ds, \tag{3.13}$$

with unknown function w instead of v' in (3.12).

Suppose that

$$u_0 \in H_0^1(\Omega) \text{ or } H^1(\Omega), \quad f \in C^\rho([0,T]; L^2(\Omega)).$$

Then $v_0 = M(0)u_0 \in D(\widetilde{A}(0))$. Hence

$$\|g'(t)\|_{L^2} \le Ct^{-1/2}.$$

With the aid of the differentiability of $\int_0^t \widetilde{U}(t,s)f(s)ds$ proved just before Theorem 2.1 it can be shown that

$$\|Q(t,s)\| \le C(t-s)^{-1/2}.$$

Hence the integral equation of (3.13) has a unique solution w satisfying

$$\|w(t)\|_{L^2} \le Ct^{-1/2}.$$

Define the function v by

$$v(t) = v_0 + \int_0^t w(s)ds.$$

Then (3.12) holds and, by integration, one obtains (3.10). Since

$$\begin{aligned}
&\|(R * v')(s') - (R * v')(s)\|_{L^2} \\
&= \Big\| \int_s^{s'} R(s', \tau)v'(\tau)d\tau + \int_0^s (R(s', \tau) - R(s, \tau))v'(\tau)d\tau \Big\|_{L^2} \\
&\le C \int_s^{s'} \|v'(\tau)\|_{L^2} d\tau + C(s' - s)^\rho \int_0^s \|v'(\tau)\|_{L^2} d\tau \\
&\le C \int_s^{s'} \tau^{-1/2} d\tau + C(s' - s)^\rho \int_0^s \tau^{-1/2} d\tau \\
&\le C(\sqrt{s'} - \sqrt{s}) + C(s' - s)^\rho s^{1/2} = C\Big\{ \frac{s' - s}{\sqrt{s'} + \sqrt{s}} + (s' - s)^\rho s^{1/2} \Big\}
\end{aligned}$$

for $0 < s < s' \le T$, $R * v'$ is locally Hölder continuous in $L^2(\Omega)$ of order ρ in $(0, T]$. Hence it follows from (3.10) that v satisfies (3.9).

Let the function u be defined by

$$u(t) = L(t)^{-1}\{f(t) + (R * f)(t) - (R * v')(t) - v'(t)\}.$$

Then according to (3.9)

$$\begin{aligned}
L(t)u(t) &= f(t) + (R * f)(t) - (R * v')(t) - v'(t) \\
&\in A(t)v(t) = L(t)M(t)^{-1}v(t). \qquad (3.14)
\end{aligned}$$

Since $L(t)$ is invertible, (3.14) implies

$$u(t) \in M(t)^{-1}v(t) \text{ or } M(t)u(t) = v(t). \qquad (3.15)$$

From the first half of (3.14) and the second half of (3.15) one derives

$$(Mu)' + Lu + R * (Mu)' = f + R * f. \qquad (3.16)$$

Convoluting K and (3.16)

$$K * (Mu)' + K * Lu + K * R * (Mu)' = K * f + K * R * f. \qquad (3.17)$$

Adding (3.16) and (3.17), and using (3.5) one obtains

$$(Mu)' + Lu + K * Lu = f.$$

Clearly,

$$\lim_{t\to 0} M(t)u(t) = \lim_{t\to 0} v(t) = v(0) = v_0 = M(0)u_0.$$

Hence u is a desired solution of the initial value problem (3.1)–(3.2).

THEOREM 3.1 *Let $f \in C^\rho([0,T]; L^2(\Omega))$ and $u_0 \in H_0^1(\Omega)$ or $u_0 \in H^1(\Omega)$ according as the boundary condition is of Dirichlet or Neumann type. Then, a solution to the problem (3.1)-(3.2) such that*

$$u \in C((0,T]; L^2(\Omega)),\ M(\cdot)u(\cdot) \in C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)),$$
$$u(t) \in D(L(t)) \text{ for } t \in (0,T],\ L(\cdot)u(\cdot) \in C((0,T]; L^2(\Omega))$$

exists and is unique.

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4 Equations in L^p spaces

In this section we consider the problem (1.1) in $L^p(\Omega)$ in case of the Dirichlet boundary condition. Assume that

$$1 < p < \frac{3}{2}. \tag{4.1}$$

Note that (4.1) implies $2 - 2/p < 1/p < 1$. Instead of (III), (IV) and (VI) in Section 1, we assume that

$$m \in C^{1,\rho}([0,T]; L^\infty(\Omega)), \quad 2 - \frac{2}{p} < \rho \le 1, \quad m \ge 0 \text{ a.e.}, \tag{4.2}$$

and

$$|D_t m(x,t)| \le C m(x,t)^\alpha, \quad 2 - \frac{2}{p} < \alpha < \frac{1}{p}. \tag{4.3}$$

The operator $M(t)$ is the multiplication by $m(x,t)$ and $L(t)$ is the realization of $\mathcal{L}(t)$ in $L^p(\Omega)$ with the Dirichlet boundary condition. Let $A(t) = L(t)M(t)^{-1}$. According to [2] the following inequality holds:

$$|\lambda| \|m(\cdot,t)^{1/p}u\|_{L^p}^p + \|u\|_{L^p}^p \le C\|f\|_{L^p}^p, \quad t \in [0,T], \tag{4.4}$$

for $(\lambda m(\cdot,t) + L(t))u = f$ and $\lambda \in \Sigma = \{\lambda \in \mathbb{C} : \text{Re}\lambda \ge -c_0(|\text{Im}\lambda| + 1)\}$. By virtue of Hölder's inequality and (4.4), for all $\lambda \in \Sigma$ and $t \in [0,T]$, one has

$$\|m(\cdot,t)^\alpha u\|_{L^p} \le \|m(\cdot,t)^{1/p}u\|_{L^p}^{\alpha p} \|u\|_{L^p}^{1-\alpha p} \le C|\lambda|^{-\alpha}\|f\|_{L^p}.$$

Hence using (4.3), for all $\lambda \in \Sigma$ and $t \in [0,T]$, one obtains

$$\|D_t m(\cdot,t)u\|_{L^p} \le C\|m(\cdot,t)^\alpha u\|_{L^p} \le C|\lambda|^{-\alpha}\|f\|_{L^p},$$

or

$$\|\dot{M}(t)(\lambda M(t)+L(t))^{-1}f\|_{L^p} \le C|\lambda|^{-\alpha}\|f\|_{L^p}. \tag{4.5}$$

Furthermore, (4.4) implies for all $\lambda \in \Sigma$, $|\lambda| \ge 1$, and $t \in [0,T]$,

$$\|M(t)(\lambda M(t)+L(t))^{-1}\| \le C|\lambda|^{-1/p}, \quad \|(\lambda M(t)+L(t))^{-1}\| \le C, \tag{4.6}$$

$$\|L(t)(\lambda M(t)+L(t))^{-1}\| \le C|\lambda|^{1-1/p}. \tag{4.7}$$

LEMMA 4.1 *For $t \in [0,T]$ and $\lambda \in \Sigma$, $|\lambda| \ge 1$,*

$$\big\|D_t(\lambda + A(t))^{-1}\big\| \le C|\lambda|^{1-\alpha-1/p}, \tag{4.8}$$

$$\begin{aligned} &\big\|D_t(\lambda + A(t))^{-1} - D_s(\lambda + A(s))^{-1}\big\| \\ &\quad \le C\big\{|t-s||\lambda|^{2-\alpha-1/p} + |t-s|^\rho|\lambda|^{1-1/p}\big\}. \end{aligned} \tag{4.9}$$

Proof. First note that

$$\begin{aligned} &D_t(\lambda + A(t))^{-1} = D_t\big\{M(t)(\lambda M(t)+L(t))^{-1}\big\} \\ &= \dot{M}(t)(\lambda M(t)+L(t))^{-1} \\ &\quad - M(t)(\lambda M(t)+L(t))^{-1}(\lambda\dot{M}(t)+\dot{L}(t))(\lambda M(t)+L(t))^{-1} \\ &= \dot{M}(t)(\lambda M(t)+L(t))^{-1} \\ &\quad - M(t)(\lambda M(t)+L(t))^{-1}\lambda\dot{M}(t)(\lambda M(t)+L(t))^{-1} \\ &\quad - M(t)(\lambda M(t)+L(t))^{-1}\dot{L}(t)L(t)^{-1}L(t)(\lambda M(t)+L(t))^{-1}. \end{aligned} \tag{4.10}$$

The inequality (4.8) is an easy consequence of (4.10), (4.5), (4.6), (4.7) and the uniform boundedness of $\dot{L}L^{-1}$.

To show (4.9) we begin by estimating the increments of the first term in the last side of (4.10). For this purpose we consider the identity:

$$\begin{aligned} &\dot{M}(t)(\lambda M(t)+L(t))^{-1} - \dot{M}(s)(\lambda M(s)+L(s))^{-1} \\ &\quad = (\dot{M}(t)-\dot{M}(s))(\lambda M(t)+L(t))^{-1} \\ &\qquad + \dot{M}(s)\big\{(\lambda M(t)+L(t))^{-1} - (\lambda M(s)+L(s))^{-1}\big\}. \end{aligned} \tag{4.11}$$

Assumption (4.2) and inequality (4.6) yield

$$\|(\dot{M}(t)-\dot{M}(s))(\lambda M(t)+L(t))^{-1}\| \le C|t-s|^\rho. \tag{4.12}$$

Using (4.5), (4.7) and noting that

$$\|(M(t)-M(s))(\lambda M(t)+L(t))^{-1}\| \leq C|t-s|, \tag{4.13}$$

for all $\lambda \in \Sigma$, $|\lambda| \geq 1$, one obtains

$$\begin{aligned}
&\|\dot{M}(s)\left\{(\lambda M(t)+L(t))^{-1}-(\lambda M(s)+L(s))^{-1}\right\}\| \\
&\quad = \|\dot{M}(s)(\lambda M(s)+L(s))^{-1}\{\lambda M(s)+L(s)-\lambda M(t)-L(t)\} \\
&\qquad\quad \times (\lambda M(t)+L(t))^{-1}\| \\
&\quad = \|\lambda\dot{M}(s)(\lambda M(s)+L(s))^{-1}(M(s)-M(t))(\lambda M(t)+L(t))^{-1} \\
&\qquad + \dot{M}(s)(\lambda M(s)+L(s))^{-1}(L(s)-L(t))L(t)^{-1}L(t)(\lambda M(t)+L(t))^{-1}\| \\
&\quad \leq C|\lambda||\lambda|^{-\alpha}|t-s| + C|\lambda|^{-\alpha}|t-s||\lambda|^{1-1/p} \\
&\quad = C|t-s|(|\lambda|^{1-\alpha}+|\lambda|^{1-\alpha-1/p}) \leq C|t-s||\lambda|^{1-\alpha}.
\end{aligned} \tag{4.14}$$

From (4.11), (4.12) and (4.14) it follows

$$\begin{aligned}
&\|\dot{M}(t)(\lambda M(t)+L(t))^{-1}-\dot{M}(s)(\lambda M(s)+L(s))^{-1}\| \\
&\leq C\{|t-s|^{\rho}+|t-s|\,|\lambda|^{1-\alpha}\}.
\end{aligned} \tag{4.15}$$

Then from (4.8) one deduces

$$\begin{aligned}
&\|M(t)(\lambda M(t)+L(t))^{-1}-M(s)(\lambda M(s)+L(s))^{-1}\| \\
&\leq \left|\int_s^t \|D_r(\lambda+A(r))^{-1}\|\,dr\right| \leq C|t-s||\lambda|^{1-\alpha-1/p}.
\end{aligned} \tag{4.16}$$

With the aid of (4.16), (4.5), (4.6) and (4.15) one obtains

$$\begin{aligned}
&\|M(t)(\lambda M(t)+L(t))^{-1}\lambda\dot{M}(t)(\lambda M(t)+L(t))^{-1} \\
&\qquad - M(s)(\lambda M(s)+L(s))^{-1}\lambda\dot{M}(s)(\lambda M(s)+L(s))^{-1}\| \\
&\quad = \|\{M(t)(\lambda M(t)+L(t))^{-1}-M(s)(\lambda M(s)+L(s))^{-1}\} \\
&\qquad \times \lambda\dot{M}(t)(\lambda M(t)+L(t))^{-1}+M(s)(\lambda M(s)+L(s))^{-1} \\
&\qquad \times \lambda\{\dot{M}(t)(\lambda M(t)+L(t))^{-1}-\dot{M}(s)(\lambda M(s)+L(s))^{-1}\}\| \\
&\quad \leq C|t-s||\lambda|^{1-\alpha-1/p}|\lambda||\lambda|^{-\alpha}+C|\lambda|^{-1/p}|\lambda|\{|t-s|^{\rho}+|t-s||\lambda|^{1-\alpha}\} \\
&\quad = C|t-s||\lambda|^{2-2\alpha-1/p}+C|t-s|^{\rho}|\lambda|^{1-1/p}+C|t-s||\lambda|^{2-\alpha-1/p} \\
&\quad \leq C\{|t-s||\lambda|^{2-\alpha-1/p}+|t-s|^{\rho}|\lambda|^{1-1/p}\}.
\end{aligned} \tag{4.17}$$

Consider now the following identity concerning the last term of (4.10):

$$\begin{aligned}
&M(t)(\lambda M(t)+L(t))^{-1}\dot{L}(t)(\lambda M(t)+L(t))^{-1} \\
&\quad - M(s)(\lambda M(s)+L(s))^{-1}\dot{L}(s)(\lambda M(s)+L(s))^{-1} \\
&= \{M(t)(\lambda M(t)+L(t))^{-1} - M(s)(\lambda M(s)+L(s))^{-1}\} \\
&\quad \times \dot{L}(t)(\lambda M(t)+L(t))^{-1} + M(s)(\lambda M(s)+L(s))^{-1} \\
&\quad \times \{\dot{L}(t)(\lambda M(t)+L(t))^{-1} - \dot{L}(s)(\lambda M(s)+L(s))^{-1}\}. \qquad (4.18)
\end{aligned}$$

In view of (4.16) the norm of the first term of the right hand side of (4.18) is estimated by

$$\begin{aligned}
&\|\{M(t)(\lambda M(t)+L(t))^{-1} - M(s)(\lambda M(s)+L(s))^{-1}\}\dot{L}(t)(\lambda M(t)+L(t))^{-1}\| \\
&\le C|t-s||\lambda|^{1-\alpha-1/p}|\lambda|^{1-1/p} = C|t-s||\lambda|^{2-\alpha-2/p}. \qquad (4.19)
\end{aligned}$$

From (4.16) and

$$\begin{aligned}
&\dot{L}(t)(\lambda M(t)+L(t))^{-1} - \dot{L}(s)(\lambda M(s)+L(s))^{-1} \\
&= \dot{L}(t)L(t)^{-1}L(t)(\lambda M(t)+L(t))^{-1} - \dot{L}(s)L(s)^{-1}L(s)(\lambda M(s)+L(s))^{-1} \\
&= (\dot{L}(t)L(t)^{-1} - \dot{L}(s)L(s)^{-1})L(t)(\lambda M(t)+L(t))^{-1} \\
&\quad + \dot{L}(s)L(s)^{-1}\{L(t)(\lambda M(t)+L(t))^{-1} - L(s)(\lambda M(s)+L(s))^{-1}\} \\
&= (\dot{L}(t)L(t)^{-1} - \dot{L}(s)L(s)^{-1})L(t)(\lambda M(t)+L(t))^{-1} \\
&+\dot{L}(s)L(s)^{-1}\{I - \lambda M(t)(\lambda M(t)+L(t))^{-1} - I + \lambda M(s)(\lambda M(s)+L(s))^{-1}\} \\
&= (\dot{L}(t)L(t)^{-1} - \dot{L}(s)L(s)^{-1})L(t)(\lambda M(t)+L(t))^{-1} \\
&\quad - \lambda\dot{L}(s)L(s)^{-1}\{M(t)(\lambda M(t)+L(t))^{-1} - M(s)(\lambda M(s)+L(s))^{-1}\}
\end{aligned}$$

one easily deduces

$$\begin{aligned}
&\|\dot{L}(t)(\lambda M(t)+L(t))^{-1} - \dot{L}(s)(\lambda M(s)+L(s))^{-1}\| \\
&\quad \le C|t-s|^{\rho}|\lambda|^{1-1/p} + C|\lambda||t-s||\lambda|^{1-\alpha-1/p} \\
&\quad \le C\{|t-s|^{\rho}|\lambda|^{1-1/p} + |t-s||\lambda|^{2-\alpha-1/p}\}. \qquad (4.20)
\end{aligned}$$

From (4.18), (4.19) and (4.20) it follows

$$\begin{aligned}
&\|M(t)(\lambda M(t)+L(t))^{-1}\dot{L}(t)(\lambda M(t)+L(t))^{-1}\\
&\quad - M(s)(\lambda M(s)+L(s))^{-1}\dot{L}(s)(\lambda M(s)+L(s))^{-1}\|\\
&\le C|t-s||\lambda|^{2-\alpha-2/p} + C|\lambda|^{-1/p}\{|t-s|^{\rho}|\lambda|^{1-1/p} + |t-s||\lambda|^{2-\alpha-1/p}\}\\
&= C|t-s||\lambda|^{2-\alpha-2/p} + C|t-s|^{\rho}|\lambda|^{1-2/p} + C|t-s||\lambda|^{2-\alpha-2/p}\\
&\le C\{|t-s||\lambda|^{2-\alpha-2/p} + |t-s|^{\rho}|\lambda|^{1-2/p}\}. \qquad (4.21)
\end{aligned}$$

With the aid of (4.10), (4.15), (4.17) and (4.21), since $|\lambda| \ge 1$, one concludes

$$\begin{aligned}
&\left\|D_t(\lambda + A(t))^{-1} - D_s(\lambda + A(s))^{-1}\right\|\\
&\le C\{|t-s|^{\rho} + |t-s||\lambda|^{1-\alpha}\} + C\{|t-s||\lambda|^{2-\alpha-1/p} + |t-s|^{\rho}|\lambda|^{1-1/p}\}\\
&\quad + C\{|t-s||\lambda|^{2-\alpha-2/p} + |t-s|^{\rho}|\lambda|^{1-2/p}\}\\
&\le C\{|t-s||\lambda|^{2-\alpha-1/p} + |t-s|^{\rho}|\lambda|^{1-1/p}\}.
\end{aligned}$$

The proof of the lemma is complete.

By virtue of (4.8) and $\alpha + 1/p - 1 > 1 - 1/p > 0$ one can construct the fundamental solution to the problem

$$\begin{aligned}
&v'(t) + A(t)v(t) \ni f(t), \quad 0 < t \le T,\\
&v(0) = v_0
\end{aligned} \qquad (4.22)$$

by the method of [4]:

$$U(t,s) = \exp(-(t-s)A(t)) + \int_s^t \exp(-(t-\tau)A(t))\Phi(\tau,s)d\tau, \quad (4.23)$$

$$\Phi(t,s) = \Phi_1(t,s) + \int_s^t \Phi_1(t,\tau)\Phi(\tau,s)d\tau, \qquad (4.24)$$

$$\Phi_1(t,s) = -(D_t + D_s)\exp(-(t-s)A(t)) \qquad (4.25)$$

$$= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}D_t(\lambda + A(t))^{-1}d\lambda. \qquad (4.26)$$

The following inequalities are simple consequences of (4.8):

$$\|\Phi_1(t,s)\| \le C(t-s)^{\alpha+1/p-2}, \quad \|\Phi(t,s)\| \le C(t-s)^{\alpha+1/p-2}. \qquad (4.27)$$

LEMMA 4.2 *For $s < \tau < t$ the following estimate holds:*

$$\|\Phi_1(t,s) - \Phi_1(\tau,s)\| \le C\Big\{(t-\tau)^{\rho}(t-s)^{1/p-2} + \frac{t-\tau}{t-s}(\tau-s)^{\alpha+1/p-2}\Big\}. \quad (4.28)$$

Proof. Consider the identity

$$\Phi_1(t,s) - \Phi_1(\tau,s)$$
$$= -\frac{1}{2\pi i}\int_\Gamma e^{\lambda(t-s)}\{D_t(\lambda + A(t))^{-1} - D_\tau(\lambda + A(\tau))^{-1}\}d\lambda$$
$$- \frac{1}{2\pi i}\int_\Gamma \{e^{\lambda(t-s)} - e^{\lambda(\tau-s)}\}D_\tau(\lambda + A(\tau))^{-1}d\lambda =: \sum_{j=1}^{2} I_j(t,\tau,s). \quad (4.29)$$

With the aid of Lemma 4.2 one gets

$$\|I_1(t,\tau,s)\| \le C\int_\Gamma e^{\mathrm{Re}\lambda(t-s)}\{(t-\tau)|\lambda|^{2-\alpha-1/p} + (t-\tau)^\rho|\lambda|^{1-1/p}\}|d\lambda|$$
$$\le C\{(t-\tau)(t-s)^{\alpha+1/p-3} + (t-\tau)^\rho(t-s)^{1/p-2}\}$$
$$\le C\{\frac{t-\tau}{t-s}(\tau-s)^{\alpha+1/p-2} + (t-\tau)^\rho(t-s)^{1/p-2}\}. \quad (4.30)$$

Using (4.8) one derives

$$\|I_2(t,\tau,s)\| = \left\|\frac{1}{2\pi i}\int_\Gamma\int_\tau^t D_r e^{\lambda(r-s)}drD_\tau(\lambda + A(\tau))^{-1}d\lambda\right\|$$
$$= \left\|\frac{1}{2\pi i}\int_\tau^t\int_\Gamma \lambda e^{\lambda(r-s)}D_\tau(\lambda + A(\tau))^{-1}d\lambda dr\right\|$$
$$\le C\int_\tau^t\int_\Gamma e^{\mathrm{Re}\lambda(r-s)}|\lambda|^{2-\alpha-1/p}|d\lambda|dr \le C\int_\tau^t (r-s)^{\alpha+1/p-3}dr$$
$$\le C\Big\{(\tau-s)^{\alpha+1/p-2} - (t-s)^{\alpha+1/p-2}\Big\}$$
$$= C(\tau-s)^{\alpha+1/p-2}\Big\{1 - \Big(\frac{\tau-s}{t-s}\Big)^{2-\alpha-1/p}\Big\}$$
$$< C(\tau-s)^{\alpha+1/p-2}\Big(1 - \frac{\tau-s}{t-s}\Big) = (\tau-s)^{\alpha+1/p-2}\frac{t-\tau}{t-s}. \quad (4.31)$$

The assertion of the lemma follows from (4.29), (4.30) and (4.31).

Arguing as in the proof of (2.11) one deduces

$$D_t\int_s^t \exp(-(t-\tau)A(t))\Phi_1(\tau,s)d\tau = -\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau$$
$$-\int_s^t D_\tau \exp(-(t-\tau)A(t))(\Phi_1(\tau,s) - \Phi_1(t,s))d\tau$$
$$+ \exp(-(t-s)A(t))\Phi_1(t,s). \quad (4.32)$$

Then making use of (4.27) and Lemma 4.2 one gets

$$\left\| \int_s^t e^{-(t-\tau)A(t)} \Phi_1(\tau, s) d\tau \right\| \leq C(t-s)^{\alpha+2/p-2}, \tag{4.33}$$

$$\left\| D_t \int_s^t e^{-(t-\tau)A(t)} \Phi_1(\tau, s) d\tau \right\| \leq C\{(t-s)^{\alpha+2/p-3} + (t-s)^{\rho+2/p-3}\}. \tag{4.34}$$

Following the proof of (2.12) one can show with the aid of (4.33), (4.34) and $\alpha + 2/p - 2 > 0$ that $\int_s^t e^{-(t-\tau)A(t)} \Phi(\tau, s) d\tau$ is differentiable with respect to t and

$$\left\| \int_s^t e^{-(t-\tau)A(t)} \Phi(\tau, s) d\tau \right\| \leq C(t-s)^{\alpha+2/p-2},$$

$$\left\| D_t \int_s^t e^{-(t-\tau)A(t)} \Phi(\tau, s) d\tau \right\| \leq C\{(t-s)^{\alpha+2/p-3} + (t-s)^{\rho+2/p-3}\}.$$

Thus, arguing as in Section 2, we establish the following theorem.

THEOREM 4.1 *Under the assumptions of the present section the fundamental solution* $U(t, s)$ *to the problem* (4.22) *exists, is represented by* (4.26) *and satisfies*

$$\|U(t, s)\| \leq C(t-s)^{1/p-1}, \quad \|D_t U(t, s)\| \leq C(t-s)^{1/p-2}, \quad 0 \leq s < t \leq T.$$

If $v_0 \in D(A(0))$ *and* $f \in C^\gamma([0, T]; L^p(\Omega))$, $\gamma > 1 - 1/p$, *then*

$$v(t) = U(t, 0)v_0 + \int_0^t U(t, s) f(s) ds$$

is the unique solution to the initial value problem (4.22).

Using Theorem 4.1 one can solve the initial value problem (1.1) in the space $L^p(\Omega)$.

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Exponential attractors for semiconductor equations

Angelo Favini, Alfredo Lorenzi and Atsushi Yagi[1]

Abstract This paper studies the asymptotic behaviour of solutions to the classical semiconductor equations due to Shockley. We will construct not only global solutions but also exponential attractors for the dynamical system determined from the Cauchy problem. Exponential attractors — such a notion was introduced by Eden, Foias, Nicolaenko and Temam — are positively invariant sets which contain the global attractor, have finite fractal dimensions and attract every trajectory in an exponential rate.

1 Introduction

We are concerned with the initial and boundary value problem for the semiconductor equations

$$\begin{cases} D_t u = a\Delta u - \mu\nabla\cdot\{u\nabla\chi\} + f(1-uv) + g(x), & \text{in } \Omega\times(0,+\infty),\\ D_t v = b\Delta v + \nu\nabla\cdot\{v\nabla\chi\} + f(1-uv) + g(x), & \text{in } \Omega\times(0,+\infty),\\ 0 = c\Delta\chi - u + v + h(x), & \text{in } \Omega\times(0,+\infty),\\ u = v = \chi = 0, & \text{on } \Gamma_D\times(0,+\infty),\\ D_n u = D_n v = D_n\chi = 0, & \text{on } \Gamma_N\times(0,+\infty),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, where $d = 2, 3$, with boundary $\partial\Omega$, n and D_n denoting, respectively, the outward unit vector normal to $\partial\Omega$ and the normal derivative.

The semiconductor equations was presented by Shockley [25] almost fifty years before to describe the flows of electrons and holes in a semiconductor. For the physical background and the details of modeling we refer to the papers

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[12, 24, 26].

The unknown functions u and v denote the densities of electrons and holes, respectively, at a position x in a semiconductor device occupying Ω and at time $t \geq 0$, electrons and holes diffuse with positive diffusion coefficients $a > 0$ and $b > 0$. The terms $-\mu\nabla \cdot \{u\nabla\chi\}$ and $\nu\nabla \cdot \{v\nabla\chi\}$ denote the drift-diffusions of electron and hole, where $\mu > 0$ and $\nu > 0$ are the positive mobilities, respectively, of electron and hole. The function χ stands for electrostatic potential and is determined by the Poisson equation, where $c > 0$ is the dielectric constant. In addition, the reaction term $f(1-uv)$ denotes the effects of generation and recombination of electrons and holes. After an appropriate normalization, electrons and holes are generated with a rate $f \geq 0$ and are recombined with a rate fuv. The functions $g \geq 0$ and h are given and both represent given external forces.

The boundary $\partial\Omega = \Gamma$ is split into two parts Γ_D and Γ_N. On Γ_D, the homogeneous Dirichlet boundary conditions are imposed on the unknown functions u and v and as well on the potential χ. On Γ_N, the Neumann boundary conditions are supposed to hold for the densities u, v and the potential χ. For electrons and holes, nonnegative initial densities $u_0 \geq 0$ and $v_0 \geq 0$ are given in Ω.

Many authors have already contributed to the study of semiconductor equations. The stationary problems were studied, e.g., by [8, 9, 15, 19, 18], [1, Chapter 6]. The existence of stationary solutions to the system (1.1) was established in various situations. On the contrary, the uniqueness of solutions is known only in a special case (cf. [1, Theorem 6.2]).

The evolution problems were analyzed, for instance, in [5, 6, 7, 10, 11, 16, 20, 21, 23]. In particular, the asymptotic behavior of solutions was studied in [7, 10, 11, 21]. Mock [21] first proved in a simple case that every solution converges to the stationary solution at an exponential rate. Gajewski [10] and Gajewski and Gröger [11] generalized this result, but they still assumed some conditions which guarantee unique solvability of the stationary problem. In the cases where the stationary solutions are possibly nonunique, Fang and Ito [7] constructed a global attractor for a dynamical system determined from the evolution problem (1.1). More precisely, they constructed a global attractor with finite Hausdorff dimension under the so called spectral gap condition for the Laplacian in Ω.

In this paper we are concerned with the existence of exponential attractors for the dynamical system determined from the evolution problem (1.1) without any particular spectral condition. The notion of exponential attractor was introduced by Eden, Foias, Nicolaenko and Temam [3] (see also [27]) as a convenient set which characterizes the longtime behavior of an infinite-dimensional dynamical system. In fact, the exponential attractor is a positively invariant compact set including the global attractor, has a finite fractal dimension and attracts every trajectory in an exponential rate.

For constructing the exponential attractors, two general methods are known. First one is due to Eden et al. [3]. Their method is based on the squeezing property of a nonlinear semigroup which is defined from the Cauchy problem. The squeezing property means that the nonlinear semigroup is a finite-rank perturbation of some contraction. Eden et al. [3] gives also some sufficient conditions for the squeezing property in the case where the nonlinear semigroups are defined from semilinear abstract evolution equations in Hilbert spaces. But, as the estimate (3.7) in Section 3 shows, our semilinear term determined from (1.1) seems too strong to fulfill the sufficient condition. The second method is due to Efendiev, Miranville and Zelik [4] which has been presented more recently. In their method, the compact Lipschitz condition for semigroup plays a principal role and the condition can be verified directly by the smoothing effect of solutions for the evolution equations, and therefore their method is available even in Banach spaces, see Section 6.

We shall show in this paper that the semigroup defined from the semiconductor (1.1) fulfills this property without assuming any spectral gap condition.

Throughout this paper, Ω denotes a bounded domain with Lipschitz boundary (see [14]) in $\mathbb{R}^d$, where $d = 2$ or 3. The boundary Γ is split into two parts Γ_D and Γ_N, and Γ_D is a nonempty open subset of Γ. Related to the splitting we assume the same sphere conditions as [1, (1.22)]:

$$|B(x_0; R) \cap \Gamma_D| \geq \gamma R^{d-1} \qquad \text{for any } x_0 \in \Gamma_D, \tag{1.2}$$

$$|B(x_0; R) \cap \Omega| \geq \gamma R^d \qquad \text{for any } x_0 \in \Gamma_N,\ B(x_0; R) \cap \Gamma_D = \emptyset \tag{1.3}$$

with some constant $\gamma > 0$, where $B(x_0; R)$ denotes an open ball centralized at x_0 with radius $R > 0$.

As interested in studying asymptotic behavior of global solutions to (1.1), we will assume for the sake of simplicity that the mobilities μ and ν of drift-diffusions are both constant and the rate f of generation and recombination is also constant. The function $g(x)$ is a given nonnegative L^2 function, i.e.,

$$0 \leq g \in L^2(\Omega), \tag{1.4}$$

and $h(x)$ is a given bounded real function, i.e.,

$$h \in L^\infty(\Omega). \tag{1.5}$$

Sections 3-6 are devoted to considering the two-dimensional problem. In Section 3, we construct a unique local solution for each pair of initial functions $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$ and show that the local solution is Lipschitz continuous with respect to the initial functions. In Section 4, we show that nonnegativity of u_0 and v_0 implies that of local solution by the truncation method. In Section 5, a priori estimates concerning the L^2 norm are obtained. In Section 6, we introduce a dynamical system determined from the problem (1.1) and construct its exponential attractors. In Section 7, we apply these techniques to the three-dimensional problem.

2 Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega = \Gamma$, where $d = 2, 3$.

For $s \geq 0$, $H^s(\Omega)$ denotes the usual Sobolev space (see [2, 14, 28]). For $0 \leq s \leq 1$, $H^s(\Omega)$ coincides with the complex interpolation space

$$H^s(\Omega) = [H^0(\Omega), H^1(\Omega)]_s \tag{2.1}$$

between $H^0(\Omega) = L^2(\Omega)$ and $H^1(\Omega)$. Indeed, this result is well known in $\mathbb{R}^d$; then, in Ω, this is verified by using the extending operator of functions in Ω to those in $\mathbb{R}^d$ which is continuous from $H^s(\Omega)$ to $H^s(\mathbb{R}^d)$ for every $0 \leq s \leq 1$ ([28, 4.2.3] or [14, Theorem 1.4.3.1]).

The space $H^s(\Omega)$, $0 < s < 1$ (when $d = 3$, $s = 1$ is included), is embedded in $L^p(\Omega)$, where $p = 2d/(d - 2s)$, with the estimate

$$\|u\|_{L^p} \leq C_s \|u\|_{H^s}, \qquad u \in H^s(\Omega),\ p = \frac{2d}{d-2s}. \tag{2.2}$$

Similarly, since those embeddings and estimates are valid in $\mathbb{R}^d$ (see [28]), these are verified by using the same extending operator mentioned above which is continuous also from $L^p(\Omega)$ to $L^p(\mathbb{R}^d)$ for every $1 < p < \infty$.

We denote by $H^1_D(\Omega)$ the space

$$H^1_D(\Omega) = \{u \in H^1(\Omega);\ u = 0 \ \text{ on } \ \Gamma_D\}.$$

Here, Γ_D is a nonempty open subset of Γ which is split into two parts Γ_D and Γ_N with conditions (1.2) and (1.3). Obviously, $H^1_D(\Omega)$ is a closed subspace of $H^1(\Omega)$. The antidual space of $H^1_D(\Omega)$ is denoted by $H^1_D(\Omega)'$.

Identifying $L^2(\Omega)$ and its antidual space, let us consider a triplet of spaces $H^1_D(\Omega) \subset L^2(\Omega) \subset H^1_D(\Omega)'$. On $H^1_D(\Omega)$, we consider a sesquilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx, \qquad u,\, v \in H^1_D(\Omega).$$

Obviously this form is continuous on $H^1_D(\Omega) \times H^1_D(\Omega)$. And, by the Poincaré inequality (e.g., see [1, (1.30)] or [2, Chap.IV, Sec.7, Remark 4]), the form is seen to be coercive on $H^1_D(\Omega)$. Then, following the usual procedure, we can define a linear isomorphism Λ from $H^1_D(\Omega)$ onto $H^1_D(\Omega)'$ which is also a sectorial linear operator of $H^1_D(\Omega)'$. In fact, Λ is characterized by

$$\langle \Lambda u, v\rangle_{H^1_D{}' \times H^1_D} = \int_\Omega \nabla u \cdot \nabla \overline{v}\, dx, \qquad u \in H^1_D(\Omega),\ v \in H^1_D(\Omega). \tag{2.3}$$

Then Λ is considered as a realization of $-\Delta$ in the space $H^1_D(\Omega)'$ under the homogeneous Dirichlet boundary conditions on Γ_D and the homogeneous Neumann boundary conditions on Γ_N.

It is known that the domain of the square root $\Lambda^{1/2}$ is characterized by

$$\mathcal{D}(\Lambda^{1/2}) = L^2(\Omega). \tag{2.4}$$

Moreover, for $1/2 \le \theta \le 1$,

$$\mathcal{D}(\Lambda^{\theta}) = [L^2(\Omega), H^1_D(\Omega)]_{2\theta-1} \subset [L^2(\Omega), H^1(\Omega)]_{2\theta-1} = H^{2\theta-1}(\Omega). \tag{2.5}$$

The operator Λ may not enjoy the optimal shift property that $\Lambda u \in L^2(\Omega)$ implies $u \in H^2(\Omega)$ due to the boundary conditions of mixed type (as studied in [17] or [14, Chap. 4]). According to Bensoussan and Frehse [1, Theorem 2.2], however, it is known that the splitting conditions (1.2) and (1.3) provide the existence of a certain exponent $p > 2$ for which $\Lambda u \in L^2(\Omega)$ implies $u \in W^1_p(\Omega)$ with the estimates

$$\|u\|_{W^1_p} \le C\|\Lambda u\|_{L^2}, \qquad u \in \Lambda^{-1}(L^2(\Omega)). \tag{2.6}$$

In the case when $d = 2$, we shall make essential use of this regularity.

Let $u \in H^2(\Omega)$ and $\chi \in H^2_N(\Omega) = \{\chi \in H^2(\Omega);\, D_n\chi = 0 \text{ on } \Gamma_N\}$. Then we easily observe that

$$\nabla \cdot \{u\nabla\chi\} = \nabla u \cdot \nabla\chi + u\Delta\chi \in L^2(\Omega)$$

and the formula

$$\langle \nabla \cdot \{u\nabla\chi\}, v\rangle_{(H^1_D)'\times H^1_D} = \int_\Omega \nabla \cdot \{u\nabla\chi\}\overline{v}\, dx$$

$$= \int_\Gamma uD_n\chi\,\overline{v}\, dx - \int_\Omega u\nabla\chi\cdot\nabla\overline{v}\, dx = -\int_\Omega u\nabla\chi\cdot\nabla\overline{v}\, dx \qquad \text{for all } v \in H^1_D(\Omega).$$

Since $\chi \in \Lambda^{-1}(L^2(\Omega))$ implies $D_n\chi = 0$ on Γ_N in an appropriate weak sense and since $u \in H^{d/p}(\Omega)$ and $\chi \in W^1_p(\Omega)$, where $d < p < \infty$, imply that

$$\int_\Omega |u\nabla\chi\nabla\overline{v}|\, dx \le \|u\|_{L^{2p/(p-2)}}\|\nabla\chi\|_{L^p}\|\nabla v\|_{L^2} \le C_p\|u\|_{H^{d/p}}\|\chi\|_{W^1_p}\|v\|_{H^1_D},$$

we are naturally led to define $\nabla\cdot\{u\nabla\chi\}$ for $u \in H^{d/p}(\Omega)$ and $\chi \in \Lambda^{-1}(L^2(\Omega)) \cap W^1_p(\Omega)$ also by the formula

$$\langle \nabla \cdot \{u\nabla\chi\}, v\rangle_{(H^1_D)'\times H^1_D} = -\int_\Omega u\nabla\chi\cdot\nabla\overline{v}\, dx \qquad \text{for all } v \in H^1_D(\Omega). \tag{2.7}$$

That is, $\nabla\cdot\{u\nabla\chi\}$ is an element of $H^1_D(\Omega)'$ with the estimate

$$\|\nabla\cdot\{u\nabla\chi\}\|_{(H^1_D)'} \le C_p\|u\|_{H^{d/p}}\|\chi\|_{W^1_p},$$
$$u \in H^{d/p}(\Omega),\ \chi \in \Lambda^{-1}(L^2(\Omega)) \cap W^1_p(\Omega), \tag{2.8}$$

where $d < p < \infty$.

Throughout the paper we shall use the following notation. The symbol $n(x)$ denotes the outward normal vector at a point $x \in \Gamma$ for which the normal vector is defined.

Let X be a Banach space and let I be an interval of $\mathbb{R}$. $\mathcal{C}(I; X)$, $\mathcal{C}^\theta(I; X)$ $(0 < \theta < 1)$ and $\mathcal{C}^1(I; X)$ denote the space of X-valued continuous functions, Hölder continuous functions with exponent θ and continuously differentiable functions defined on I, respectively. When I is a bounded closed interval, these are Banach spaces endowed with the usual norms.

---

3 Local solutions

In this section we shall construct local solutions to (1.1) in the two-dimensional case.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary which is split into two parts Γ_D and Γ_N satisfying conditions (1.2) and (1.3).

Our goal is to apply the abstract result of [22, Theorem 3.1]. As an underlying space we take the product $H_D^1(\Omega)'$ space:

$$X = \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} : \ \varphi \in H_D^1(\Omega)' \ \text{ and } \ \psi \in H_D^1(\Omega)' \right\}. \tag{3.1}$$

Let Λ be a realization of the Laplace operator $-\Delta$ in the space $H_D^1(\Omega)'$ given by (2.6). As noticed, Λ is an isomorphism from $H_D^1(\Omega)$ onto $H_D^1(\Omega)'$. It is a sectorial operator of $H_D^1(\Omega)'$ and its fractional powers have the domains included in the Sobolev spaces as described in (2.5). The shift property (2.6) is true.

Using the operator Λ, we formulate (1.1) as the Cauchy problem for a semilinear abstract evolution equation

$$\begin{cases} D_t U + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases} \tag{3.2}$$

in the space X. Here, A is a linear operator in X given by

$$AU = \begin{pmatrix} a\Lambda & 0 \\ 0 & b\Lambda \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A)$$

with domain

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \ u \in H_D^1(\Omega) \ \text{ and } \ v \in H_D^1(\Omega) \right\}. \tag{3.3}$$

The operator $F(U)$ is a nonlinear operator in X given by

$$F(U) = \begin{pmatrix} -\mu\nabla\cdot\{u\nabla(c\Lambda)^{-1}(-u+v+h(x))\}+f(1-uv)+g(x) \\ \nu\nabla\cdot\{v\nabla(c\Lambda)^{-1}(-u+v+h(x))\}+f(1-uv)+g(x) \end{pmatrix},$$

$$U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(F)$$

with domain

$$\mathcal{D}(F) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in H^{2/p}(\Omega) \text{ and } v \in H^{2/p}(\Omega) \right\},$$

where $p > 2$ is a fixed number in such a way that (2.6) holds. In view of (2.6), (2.7) and (2.8), $F(U)$ is well defined. Let $\eta = (1/2) + (1/p) < 1$. Then, since $\mathcal{D}(\Lambda^\eta) \subset H^{2/p}(\Omega)$ due to (2.5), we have

$$\mathcal{D}(A^\eta) \subset \mathcal{D}(F). \tag{3.4}$$

We can now apply [22, Theorem 3.1] to the present problem with $\alpha = 1/2$ and $\eta = (1/2) + (1/p)$. In fact, A satisfies [22, (A)] with any $0 < \phi < \pi/2$. From (2.8),

$$\begin{cases} \|\nabla\cdot\{u\nabla\Lambda^{-1}(-u+v+h(x))\}\|_{(H_D^1)'} \le C\|u\|_{H^{2/p}}\|-u+v+h\|_{L^2} \\ \|\nabla\cdot\{v\nabla\Lambda^{-1}(-u+v+h(x))\}\|_{(H_D^1)'} \le C\|v\|_{H^{2/p}}\|-u+v+h\|_{L^2}. \end{cases} \tag{3.5}$$

Similarly,

$$\begin{aligned} \|uv\|_{(H_D^1)'} &= \sup_{\|w\|_{H_D^1}\le 1} \left| \int_\Omega uvw\,dx \right| \\ &\le C \sup_{\|w\|_{H_D^1}\le 1} \|u\|_{L^{2p/(p-2)}}\|v\|_{L^2}\|w\|_{L^p} \le C\|u\|_{H^{2/p}}\|v\|_{L^2}. \end{aligned} \tag{3.6}$$

Therefore, on account of (2.5) and (3.4), we deduce that

$$\begin{aligned} \|F(U)-F(V)\| \le C\{&(\|A^\beta U\|+\|A^\beta V\|+1)\|A^\eta(U-V)\| \\ &+(\|A^\eta U\|+\|A^\eta V\|+1)\|A^\beta(U-V)\|\}, \qquad U, V \in \mathcal{D}(A^\eta). \end{aligned} \tag{3.7}$$

This shows that [22, (F)] is fulfilled.

Since, due to (2.4),

$$\mathcal{D}(A^\beta) = \mathcal{D}(A^{1/2}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : f \in L^2(\Omega) \quad \text{and} \quad g \in L^2(\Omega) \right\}, \tag{3.8}$$

the following local existence result is deduced directly from [22, Theorem 3.1].

THEOREM 3.1 *For any pair of initial data $u_0 \in L^2(\Omega)$ and $v_0 \in L^2(\Omega)$, there exists a unique local solution $U = \begin{pmatrix} u \\ v \end{pmatrix}$ to (3.2) in the function space*

$$u,\, v \in \mathcal{C}((0, T_{U_0}];\, H^1_D(\Omega)) \cap \mathcal{C}([0, T_{U_0}];\, L^2(\Omega)) \cap \mathcal{C}^1((0, T_{U_0}];\, H^1_D(\Omega)'), \quad (3.9)$$

here $T_{U_0} > 0$ only depends on the sum of norms $\|u_0\|_{L^2} + \|v_0\|_{L^2}$. Moreover, the estimate

$$\sqrt{t}(\|u(t)\|_{H^1_D} + \|v(t)\|_{H^1_D}) + \|u(t)\|_{L^2} + \|v(t)\|_{L^2} \le C_{U_0}, \qquad 0 < t \le T_{U_0},$$

holds, where $C_{U_0} > 0$ only depends on the sum of norms $\|u_0\|_{L^2} + \|v_0\|_{L^2}$ as well.

Furthermore, consider a closed ball of initial functions

$$B_R = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix};\ u_0 \in L^2(\Omega) \text{ and } v_0 \in L^2(\Omega) \right.$$
$$\left. \text{with } \|u_0\|_{L^2} + \|v_0\|_{L^2} \le R \right\},$$

where $0 < R < \infty$. For each $U_0 \in B_R$, there exists a unique local solution to (3.2) at least on a fixed interval $[0, T_R]$, $T_R > 0$ only depending on R. We can then easily obtain the following Lipschitz continuity of local solutions with respect to initial functions (cf. [22, Corollary 3.2]).

THEOREM 3.2 *Let U_0 and $\widetilde{U}_0$ be in B_R and let U and $\widetilde{U}$ be the local solutions to (3.2) on the interval $[0, T_R]$ with the initial values U_0 and $\widetilde{U}_0$ respectively. Then,*

$$\sqrt{t}\|U(t) - \widetilde{U}(t)\|_{H^1_D} + \|U(t) - \widetilde{U}(t)\|_X \le C_R \|U_0 - \widetilde{U}_0\|_X,$$
$$0 \le t \le T_R, \quad (3.10)$$

where $C_R > 0$ is a constant determined by R alone.

---

4 Nonnegativity of solutions

We now prove that the nonnegativity of the initial data implies that the local solution (cf. Theorem 3.1) is nonnegative as well.

THEOREM 4.1 *Let $0 \le u_0 \in L^2(\Omega)$ and $0 \le v_0 \in L^2(\Omega)$. Then the local solution U obtained in Theorem 3.1 also satisfies $u(t) \ge 0$ and $v(t) \ge 0$ for every $0 < t \le T_{U_0}$.*

Proof. Let $\overline{U} = \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}$ be the complex conjugate of U. Then it is clear that $\overline{U}$ is also a local solution to the same problem (3.2); this means that the local solution U is real-valued.

In order to verify $u(t) \geq 0$ and $v(t) \geq 0$, we will use the truncation method (cf. [13, Theorem 7.8]). Before using the method, however, we need to introduce approximate linear problems.

It is not difficult to construct sequences of Hölder continuous functions with values in $L^\infty(\Omega)$ such that

$$u_k,\ v_k \in \mathcal{C}^\mu([0, T_{U_0}];\ L^\infty(\Omega)), \quad 0 < \mu < 1 \tag{4.1}$$

which converge to $u(t)$ and $v(t)$ respectively as $k \to \infty$ in the space $\mathcal{C}([0, T_{U_0}];\ L^2(\Omega))$. Indeed, since u and v are L^2 valued continuous functions on $[0, T_{U_0}]$, such approximate sequences can be constructed by the cutoff of L^2 functions and the mollifier acting on the variable t.

Using u_k and v_k, we next consider the linear problem

$$\begin{cases} D_t U_k + A\widetilde{U}_k = B_k(t)\widetilde{U}_k + F, & 0 < t \leq T_{U_0}, \\ \widetilde{U}_k(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \end{cases} \tag{4.2}$$

in X. Here, $B_k(t)$ is a family of closed linear operators given by

$$B_k(t)\widetilde{U}_k = \begin{pmatrix} -\mu\nabla \cdot \{\widetilde{u}_k \nabla\chi_k(t)\} - \widetilde{u}_k v_k(t)) \\ \nu\nabla \cdot \{\widetilde{v}_k \nabla\chi_k(t)\} - u_k(t)\widetilde{v}_k \end{pmatrix}, \qquad \widetilde{U}_k = \begin{pmatrix} \widetilde{u}_k \\ \widetilde{v}_k \end{pmatrix} \in \mathcal{D}(B_k(t)),$$

where $\chi_k(t) = (c\Lambda)^{-1}(-u_k(t) + v_k(t) + h)$, with domain $\mathcal{D}(B_k(t)) \equiv \mathcal{D}(A^\eta)$, while F is an element of X defined by

$$F(x) = \begin{pmatrix} f + g(x) \\ f + g(x) \end{pmatrix}.$$

Since $\mathcal{D}(A) \subset \mathcal{D}(A^\eta) \equiv \mathcal{D}(B_k(t))$, the operators $B_k(t)$ are weak linear perturbations of A; in addition, they enjoy regularity

$$\|\{B_k(t) - B_k(s)\}A^{-\eta}\|_{\mathcal{L}(X)} \leq C_k|t - s|^\mu, \qquad 0 \leq s,\ t \leq T_{U_0}$$

due to (3.5) and (3.6). Then, by the theory of linear abstract evolution equations, it is deduced that the problem (4.2) possesses a unique solution $\widetilde{U}_k$ such that

$$\widetilde{U}_k \in \mathcal{C}((0, T_{U_0}];\ \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}];\ \mathcal{D}(A^{1/2}) \cap \mathcal{C}^1((0, T_{U_0}];\ X).$$

Furthermore, after direct calculations we verify that

$$t^\eta\|A^\eta\{\widetilde{U}_k(t) - U(t)\}\|_X \leq C_k \sup_{0\leq t\leq T_{U_0}} \|A^\beta\{U_k(t) - U(t)\}\|_X,$$

that is, as $k \to \infty$, $\widetilde{U}_k(t)$ converges to $U(t)$ for every $t \in (0, T_{U_0}]$ in $\mathcal{D}(A^\eta)$.

Therefore if we show that $\widetilde{u}_k(t) \ge 0$ and $\widetilde{v}_k(t) \ge 0$, then it follows that $u(t) \ge 0$ and $v(t) \ge 0$.

Consider a $\mathcal{C}^1$ function $H(u)$ of u such that $H(u) = u^2/2$ for $-\infty < u < 0$ and $H(u) = 0$ for $0 \le u < \infty$. We set the nonnegative function

$$\rho(t) = \int_\Omega H(\widetilde{u}_k(x,t))dx, \qquad 0 \le t \le T_{U_0}.$$

Since $H'(\widetilde{u}_k(t)) \in H^1_D(\Omega)$ (due to [13, Theorem 7.8]), it follows from

$$\rho(t+h) - \rho(t) = \left(\widetilde{u}_k(t+h) - \widetilde{u}_k(t), \int_0^1 H'(\theta\widetilde{u}_k(t+h) + (1-\theta)\widetilde{u}_k(t))d\theta\right)_{L^2}$$

that

$$\rho'(t) = \left\langle \frac{\partial \widetilde{u}_k}{\partial t}, H'(\widetilde{u}_k)\right\rangle_{(H^1_D)' \times H^1_D}.$$

Furthermore, since $\nabla H'(\widetilde{u}_k(t)) = H''(\widetilde{u}_k(t))\nabla\widetilde{u}_k(t)$ (due to [13, Theorem 7.8]), it follows that

$$\rho'(t) = -a\int_\Omega H''(\widetilde{u}_k)|\nabla\widetilde{u}_k|^2 dx$$
$$+\mu\int_\Omega \widetilde{u}_k\nabla H'(\widetilde{u}_k)\cdot\nabla\chi_k dx - f\int_\Omega v_k\widetilde{u}_k H'(\widetilde{u}_k)dx + \int_\Omega (f+g(x))H'(\widetilde{u}_k)dx.$$

Here we use Lemma 4.1 below to obtain

$$\int_\Omega \widetilde{u}_k\nabla H'(\widetilde{u}_k)\cdot\nabla\chi_k dx = \int_\Omega H'(\widetilde{u}_k)\nabla H'(\widetilde{u}_k)\cdot\nabla\chi_k dx$$
$$= \frac{1}{2c}\int_\Omega (-u_k + v_k + h)H'(\widetilde{u}_k)^2 dx.$$

Since $H'(u) \le 0$, $H''(u) \ge 0$ and $f + g(x) \ge 0$, we verify that

$$\rho'(t) \le \frac{\mu}{2c}\int_\Omega (-u_k + v_k + h)H'(\widetilde{u}_k(t))^2 dx - f\int_\Omega v_k H'(\widetilde{u}_k(t))^2 dx.$$

Furthermore, by (1.5) and (4.1),

$$\rho'(t) \le C_k\int_\Omega H'(\widetilde{u}_k)^2 dx \le C_k\rho(t), \qquad 0 < t \le T_{U_0}.$$

In this way we have shown that $\rho(t) \le \rho(0)e^{C_k t} = 0$ for every $0 < t \le T_{U_0}$ and hence $u_k(t) \ge 0$. It is the same for $v_k(t)$.

LEMMA 4.1 *For $u \in H^1_D(\Omega)$ and $\chi \in \Lambda^{-1}(L^2(\Omega)) \cap W^1_p(\Omega)$,*

$$\int_\Omega u\nabla u\cdot\nabla\chi\, dx = \tfrac{1}{2}\int_\Omega u^2\Lambda\chi\, dx. \tag{4.3}$$

Proof. If $u \in H^1_D(\Omega) \cap L^\infty(\Omega)$, then $u^2 \in H^1_D(\Omega)$. Therefore, using $2u\nabla u = \nabla(u^2)$, we verify (4.3) directly from the definition (2.3). For general $u \in H^1_D(\Omega)$, it is sufficient to approximate u by cutoff functions $\Psi_k(u) \in H^1_D(\Omega) \cap L^\infty(\Omega)$, where $\Psi_k(u) = -k$, $u \le -k$; $\Psi_k(u) = u$, $|u| \le k$; $\Psi_k(u) = k$, $u \ge k$. Owing to [13, Theorem 7.8], we conclude the desired equality.

---

5 A priori estimates and global solutions

In this section we shall establish *a priori* estimates of local solutions, which will then guarantee the existence of global solutions.

PROPOSITION 5.1 *Let* $0 \le u_0 \in L^2(\Omega)$ *and* $0 \le v_0 \in L^2(\Omega)$. *Let* $U = \begin{pmatrix} u \\ v \end{pmatrix}$ *be any nonnegative local solution to* (3.2) *on an interval* $[0, T_U]$ *such that*

$$0 \le u \in \mathcal{C}((0,T_U];\, H^1_D(\Omega)) \cap \mathcal{C}([0,T_U];\, L^2(\Omega)) \cap \mathcal{C}^1((0,T_U];\, H^1_D(\Omega)'),$$

$$0 \le v \in \mathcal{C}((0,T_U];\, H^1_D(\Omega)) \cap \mathcal{C}([0,T_U];\, L^2(\Omega)) \cap \mathcal{C}^1((0,T_U];\, H^1_D(\Omega)').$$

Then, with some constant $C > 0$ *independent of* U, *the estimate*

$$\|u(t)\|^2_{L^2} + \|v(t)\|^2_{L^2} \le C(\|u_0\|^2_{L^2} + \|v_0\|^2_{L^2} + 1), \qquad 0 \le t \le T_U, \tag{5.1}$$

holds.

Proof. Consider the duality product of $H^1_D(\Omega)' \times H^1_D(\Omega)$ between the first equation of (3.2) and u. Then, by (2.3) and (2.7),

$$\frac{1}{2}D_t\int_\Omega u^2 dx + a\int_\Omega |\nabla u|^2 dx - \mu\int_\Omega u\nabla u\cdot\nabla\chi\, dx + f\int_\Omega u^2 v\, dx = \int_\Omega (f + g(x))u\, dx,$$

where $\chi = (c\Lambda)^{-1}(-u + v + h(x))$.

Similarly, from the second equation of (3.2),

$$\frac{1}{2}D_t\int_\Omega v^2 dx + b\int_\Omega |\nabla v|^2 dx + \nu\int_\Omega v\nabla v\cdot\nabla\chi\, dx + f\int_\Omega uv^2 dx = \int_\Omega (f + g(x))v\, dx.$$

We sum up these two equalities after multiplying the first one by ν and the second one by μ, respectively. Then, since it follows from Lemma 4.1 that

$$\begin{aligned}\mu\nu\int_\Omega(-u\nabla u\cdot\nabla\chi + v\nabla v\cdot\nabla\chi)dx &= \frac{\mu\nu}{2}\int_\Omega (v^2 - u^2)\Lambda\chi\, dx \\ &= \frac{\mu\nu}{2c}\int_\Omega (u-v)^2(u+v)dx + \frac{\mu\nu}{2c}\int_\Omega h(x)(v^2 - u^2)dx,\end{aligned}$$

we have

$$\frac{1}{2}D_t\int_\Omega(\nu u^2+\mu v^2)dx+\int_\Omega(a\nu|\nabla u|^2+b\mu|\nabla v|^2)dx+\frac{\mu\nu}{2c}\int_\Omega(u-v)^2(u+v)dx$$
$$+f\int_\Omega(\nu u+\mu v)uv\,dx=\int_\Omega(f+g(x))(\nu u+\mu v)dx+\frac{\mu\nu}{2c}\int_\Omega h(x)(u^2-v^2)dx.$$

Furthermore, by (1.4),

$$\int_\Omega(f+g(x))(\nu u+\mu v)dx\le\|f+g\|_{L^2}\|\mu u+\nu v\|_{L^2}\le\varepsilon(\|u\|_{L^2}^2+\|v\|_{L^2}^2)+C_\varepsilon$$

with an arbitrary positive number $\varepsilon>0$ and a constant $C_\varepsilon>0$ depending on ε. Similarly, by (1.5),

$$\int_\Omega h(x)(u^2-v^2)dx\le\|h\|_{L^\infty}\int_\Omega\{\varepsilon(u-v)^2(u+v)+C_\varepsilon(u+v)\}dx$$
$$\le\|h\|_{L^\infty}\int_\Omega\{\varepsilon(u-v)^2(u+v)+\varepsilon(u^2+v^2)+C_\varepsilon\}dx$$

with an arbitrary positive number $\varepsilon>0$ and a constant $C_\varepsilon>0$ depending on ε. By the Poincaré inequality,

$$\int_\Omega(a\nu|\nabla u|^2+b\mu|\nabla v|^2)dx\ge\alpha\int_\Omega(u^2+v^2)dx$$

for some positive number $\alpha>0$. Therefore, taking ε sufficiently small, we obtain that

$$\frac{1}{2}D_t\int_\Omega(\nu u^2+\mu v^2)dx+\frac{\alpha}{2}\int_\Omega(u^2+v^2)dx\le C$$

for some constant $C>0$. Hence,

$$\|u(t)\|_{L^2}^2+\|v(t)\|_{L^2}^2\le C[e^{-\delta t}(\|u_0\|_{L^2}^2+\|v_0\|_{L^2}^2)+1],\qquad 0\le t\le T_U\quad(5.2)$$

for some positive exponent $\delta>0$ and some constant $C>0$.

We can now state the global existence of solutions. As the proof is quite standard, it is omitted.

THEOREM 5.1 *For any pair of initial data $0\le u_0\in L^2(\Omega)$ and $0\le v\in L^2(\Omega)$, (3.2) possesses a unique global solution in the function space*

$$0\le u\in\mathcal{C}((0,\infty);\,H_D^1(\Omega))\cap\mathcal{C}([0,\infty);\,L^2(\Omega))\cap\mathcal{C}^1((0,\infty);\,H_D^1(\Omega)'),$$
$$0\le v\in\mathcal{C}((0,\infty);\,H_D^1(\Omega))\cap\mathcal{C}([0,\infty);\,L^2(\Omega))\cap\mathcal{C}^1((0,\infty);\,H_D^1(\Omega)').$$

6 Exponential attractors

In this section we shall define a dynamical system determined from the two-dimensional semiconductor equations (1.1) and shall show that it has exponential attractors. We begin with reviewing some known results for exponential attractors.

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a compact subset of X, $\mathcal{X}$ being a metric space with the distance $d(\cdot,\cdot)$ induced from $\|\cdot\|_X$. Let $S(t)$, $0 \le t < \infty$, be a nonlinear semigroup acting on $\mathcal{X}$. Let $S(t)$ be continuous in the sense that the mapping $(t, U_0) \mapsto S(t)U_0$ is continuous from $[0,\infty)\times\mathcal{X}$ to $\mathcal{X}$. Then $S(t)$ defines a dynamical system $(S(t), \mathcal{X}, X)$ in X with the compact phase space $\mathcal{X}$.

As the phase space is compact, it is easily seen that the set

$$\mathcal{A} = \bigcap_{0\le t<\infty} S(t)\mathcal{X}$$

is the global attractor of $(S(t), \mathcal{X}, X)$. Namely, $\mathcal{A}$ is a strictly invariant set, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $t > 0$, and attracts all the trajectories in the sense that $\lim_{t\to+\infty} h(S(t)\mathcal{X}, \mathcal{A}) = 0$, where $h(B_0, B_1) = \sup_{U\in B_0} d(U, B_1) = \sup_{U\in B_0} \inf_{V\in B_1} d(U, V)$ denotes the Hausdorff semidistance of two subsets B_0 and B_1 of $\mathcal{X}$.

The exponential attractor is then defined as follows. A set $\mathcal{M}$ such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$ is called an exponential attractor of $(S(t), \mathcal{X}, X)$ (see Eden, Foias, Nicolaenko and Temam [3]) if: i) $\mathcal{M}$ is a compact subset of X with finite fractal dimension; ii) $\mathcal{M}$ is an invariant set, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$ for every $t > 0$; and iii) there exist some exponent $\delta > 0$ and a constant $C_0 > 0$ such that

$$h(S(t)\mathcal{X}, \mathcal{M}) \le C_0 e^{-\delta t}, \qquad 0 \le t < \infty. \tag{6.1}$$

Concerning the construction of exponential attractors we present a method due to Efendiev, Miranville and Zelik [4]. Assume the following two conditions. There exists a Banach space Z that is compactly embedded in X and a time $t^* > 0$ such that the operator $S(t^*)$ satisfies a Lipschitz condition of the form

$$\|S(t^*)U_0 - S(t^*)V_0\|_Z \le L_1\|U_0 - V_0\|_X, \quad U_0,\, V_0 \in \mathcal{X} \tag{6.2}$$

with a constant $L_1 > 0$. The mapping $(t, U_0) \mapsto S(t)U_0$ from $[0, t^*] \times \mathcal{X}$ into $\mathcal{X}$ also satisfies the usual Lipschitz condition

$$\|S(t)U_0 - S(s)V_0\|_X \le L_2\{|t-s| + \|U_0 - V_0\|_X\},$$
$$t,\, s \in [0, t^*],\ U_0,\, V_0 \in \mathcal{X}. \tag{6.3}$$

Then, the theorem on construction of exponential attractors in [4] jointed with [3, Theorem 3.1] provides the following theorem.

THEOREM 6.1 *If* (6.2) *and* (6.3) *are satisfied, then the dynamical system* $(S(t), \mathcal{X}, X)$ *possesses a family of exponential attractors* $\mathcal{M}$.

We will now apply the present theorem to our problem (1.1). Let X be the product space $H^1_D(\Omega)' \times H^1_D(\Omega)'$ given by (3.1), its norm being denoted by $\|\cdot\|_X$. We use also the product spaces $\mathcal{D}(A^{1/2})$ and $\mathcal{D}(A)$, which are given by (3.8) and (3.3) respectively. We equip $\mathcal{D}(A^{1/2})$ and $\mathcal{D}(A)$ with the product L^2-norm and the product H^1_D-norm, respectively, these norms being denoted by $\|\cdot\|_{\mathcal{D}(A^{1/2})}$ and $\|\cdot\|_{\mathcal{D}(A)}$.

We first introduce a set of initial functions given by

$$\mathcal{K} = \left\{ U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix};\ 0 \le u_0 \in L^2(\Omega) \text{ and } 0 \le v_0 \in L^2(\Omega) \right\}. \tag{6.4}$$

Obviously, $\mathcal{K}$ is a closed subset of the product Hilbert space $\mathcal{D}(A^{1/2})$.

Theorem 5.1 then shows that we can define a nonlinear semigroup acting on $\mathcal{K}$ from the problem (3.2) by setting

$$S(t)U_0 = U(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \qquad U_0 \in \mathcal{K},$$

where U denotes the unique global solution to (3.2).

We now notice the fact that estimates (5.2), which have been established for the local solutions, actually hold for the global solutions. This fact then shows implicitly the existence of an absorbing set of $S(t)$. In fact, let C^* be the constant appearing in (5.2) and consider the subset

$$\mathcal{B} = \{U \in \mathcal{K};\ \|U\|_{\mathcal{D}(A^{1/2})} \le \sqrt{2C^*}\} \subset \mathcal{K}. \tag{6.5}$$

Then this subset is an absorbing set in the sense that for any bounded set B of $\mathcal{K}$, there exists a time $t_B > 0$ such that $S(t)B \subset \mathcal{B}$ for all $t \ge t_B$.

This means that the asymptotic behavior of global solutions to (1.1) is reduced to that of solutions starting from $\mathcal{B}$. Moreover, since $\mathcal{B}$ is also one of bounded sets of $\mathcal{K}$, all the solutions starting from $\mathcal{B}$ themselves enter $\mathcal{B}$ after some fixed time $t_{\mathcal{B}} > 0$. In view of this fact we are led to introduce a new phase space

$$\mathcal{X} = \overline{\bigcup_{t \ge t_{\mathcal{B}}} S(t)\mathcal{B}} \qquad \text{(closure in the norm of } X\text{)}. \tag{6.6}$$

Several propositions proved below will show some nice properties of $\mathcal{X}$.

PROPOSITION 6.1 *The set* $\mathcal{X}$ *is a compact set of* X *such that* $\mathcal{X} \subset \mathcal{B}$, *and is an absorbing set of* $S(t)$.

Proof. As $\mathcal{D}(A^{1/2})$ is a separable Hilbert space, $\mathcal{B}$ is a sequentially weakly compact set of $\mathcal{D}(A^{1/2})$; therefore, $\mathcal{B}$ is a closed set of X. So the relation

$\bigcup_{t\geq t_{\mathcal{B}}} S(t)\mathcal{B} \subset \mathcal{B}$ implies $\mathcal{X} \subset \overline{\mathcal{B}} = \mathcal{B}$. In addition, since $\mathcal{B}$ is a compact set of X, it is the same for $\mathcal{X}$.

Since $\mathcal{B}$ is an absorbing set, it is the same for the set $\mathcal{X}$ by definition (6.6).

PROPOSITION 6.2 *The set $\mathcal{X}$ is a closed bounded set of $\mathcal{D}(A^{1/2})$ as well as of $\mathcal{D}(A)$.*

Proof. Since $\mathcal{X} \subset \mathcal{B}$, $\mathcal{X}$ is a bounded set of $\mathcal{D}(A^{1/2})$. It is clear that $\mathcal{X}$ is closed in $\mathcal{D}(A^{1/2})$.

In order to verify $\mathcal{X} \subset \mathcal{D}(A)$, we use the estimate established in Theorem 3.1. For any $0 < R < \infty$, there exist time $T_R > 0$ and a constant $C_R > 0$ such that

$$\|S(t)U_0\|_{\mathcal{D}(A)} \leq C_R t^{-1/2}, \quad 0 < t \leq T_R, \qquad \|U_0\|_{\mathcal{D}(A^{1/2})} \leq R.$$

Let us use this with $R = R^* = \sqrt{C^*(2C^*+1)}$. By (5.1), we observe that

$$\sup_{0\leq t<\infty} \|S(t)U_0\|_{\mathcal{D}(A^{1/2})} \leq \sqrt{C^*(2C^*+1)} = R^*, \qquad U_0 \in \mathcal{B}. \tag{6.7}$$

So, let $T^* > 0$ be a fixed time in such a way that $T^* \leq t_{\mathcal{B}}$ and $T^* \leq T_{R^*}$.

If $U_1 \in \bigcup_{t\geq t_{\mathcal{B}}} S(t)\mathcal{B}$, namely $U_1 = S(t_0)U_0$ with some $t_0 \geq t_{\mathcal{B}}$ and some $U_0 \in \mathcal{B}$, then

$$\|U_1\|_{\mathcal{D}(A)} = \|S(T^*)S(t_0 - T^*)U_0\|_{\mathcal{D}(A)} \leq C_{R^*}(T^*)^{-1/2},$$

because of $\|S(t_0 - T^*)U_0\|_{\mathcal{D}(A^{1/2})} \leq R^*$. Thus we have proved that

$$\bigcup_{t\geq t_{\mathcal{B}}} S(t)\mathcal{B} \subset \{U \in \mathcal{D}(A);\ \|U\|_{\mathcal{D}(A)} \leq C_{R^*}(T^*)^{-1/2}\}.$$

As any closed bounded ball of $\mathcal{D}(A)$ is sequentially weakly compact, it is a closed set of X. Therefore we deduce that

$$\mathcal{X} \subset \{U \in \mathcal{D}(A);\ \|U\|_{\mathcal{D}(A)} \leq C_{R^*}(T^*)^{-1/2}\}. \tag{6.8}$$

It is clear that $\mathcal{X}$ is closed in $\mathcal{D}(A)$.

PROPOSITION 6.3 *The set $\mathcal{X}$ is an invariant set of $S(t)$, i.e., $S(t)\mathcal{X} \subset \mathcal{X}$ for every $t > 0$.*

Proof. Let us apply Theorem 3.2 with $R = R^* = \sqrt{C^*(2C^*+1)}$ to obtain that

$$\|S(t)U_0 - S(t)V_0\|_X \leq C_{R^*}\|U_0 - V_0\|_X,$$
$$0 \leq t \leq T_{R^*},\ \|U_0\|_{\mathcal{D}(A^{1/2})} \leq R^*,\ \|V_0\|_{\mathcal{D}(A^{1/2})} \leq R^*. \tag{6.9}$$

From this we deduce that

$$\|S(t)U_0 - S(t)V_0\|_X \le (C_{R^*})^j \|U_0 - V_0\|_X, \qquad (j-1)T_{R^*} \le t \le jT_{R^*},\ U_0,\, V_0 \in \mathcal{B} \quad (6.10)$$

for every $j = 1, 2, 3, \ldots$. Indeed, from (6.7) and (6.9), this holds for $j = 1$. Assume that (6.10) holds for j. If $jT_{R^*} \le t \le (j+1)T_{R^*}$, then $0 \le t - jT_{R^*} \le T_{R^*}$. Since $\|S(jT_{R^*})U_0\|_{\mathcal{D}(A^{1/2})} \le R^*$ and $\|S(jT_{R^*})V_0\|_{\mathcal{D}(A^{1/2})} \le R^*$ (due to (6.7)), we observe that

$$\begin{aligned}\|S(t)U_0 - S(t)V_0\|_X &= \|S(t-jT_{R^*})S(jT_{R^*})U_0 - S(t-jT_{R^*})S(jT_{R^*})V_0\|_X \\ &\le C_{R^*}\|S(jT_{R^*})U_0 - S(jT_{R^*})V_0\|_X \le (C_{R^*})^{j+1}\|U_0 - V_0\|_X.\end{aligned}$$

This shows that (6.10) holds for $j+1$ also.

Therefore, for any $t \ge 0$, the operator $S(t) : \mathcal{X} \to X$ is continuous with respect to the X norm. We then observe that

$$S(t)\mathcal{X} = S(t)\overline{\bigcup_{\tau \ge t_{\mathcal{B}}} S(\tau)\mathcal{B}} \subset \overline{S(t)\bigcup_{\tau \ge t_{\mathcal{B}}} S(\tau)\mathcal{B}} \subset \mathcal{X}.$$

Hence, $S(t)$ maps $\mathcal{X}$ into itself for every $t > 0$.

PROPOSITION 6.4 *The mapping $(t, U) \mapsto S(t)U$ is locally Lipschitz continuous from $[0, \infty) \times \mathcal{X}$ into $\mathcal{X}$ in a sense that, for any $0 < T < \infty$, there exists a constant $C_T > 0$ such that*

$$\|S(t)U_0 - S(s)V_0\|_X \le C_T\{|t-s| + \|U_0 - V_0\|_X\}, \qquad t,\, s \in [0, T],\ U_0,\, V_0 \in \mathcal{X}.$$

Proof. Write

$$S(t)U_0 - S(s)V_0 = \{S(t)U_0 - S(s)U_0\} + \{S(s)U_0 - S(s)V_0\}.$$

By (6.10) we already know that

$$\|S(s)U_0 - S(s)V_0\|_X \le C_T\|U_0 - V_0\|_X.$$

In the meantime, for $0 < s < t < T$,

$$\|S(t)U_0 - S(s)U_0\|_X = \left\|\int_s^t \frac{dS(\tau)U_0}{d\tau}\, d\tau\right\|_X \le \int_s^t \|AS(\tau)U_0 + F(S(\tau)U_0)\|_X d\tau.$$

Therefore, Propositions 6.2 and 6.3 yield that

$$\|S(t)U_0 - S(s)U_0\|_X \le C(t-s).$$

We have thus constructed a dynamical system $(S(t), \mathcal{X}, X)$ in which the phase space $\mathcal{X}$ is a compact set of X and absorbs every solution starting from $\mathcal{K}$ in finite time. Since the phase space $\mathcal{X}$ is compact, the set $\mathcal{A} = \bigcap_{0\le t<+\infty} S(t)\mathcal{X}$ is nonempty and gives the global attractor of the dynamical system.

We are now in a position to apply Theorem 6.1 by verifying conditions (6.2) and (6.3). We apply Theorem 3.2 with $R = \sqrt{2C^*}$. As $\mathcal{X} \subset \mathcal{B}$,

$$\|S(t)U_0 - S(t)V_0\|_{\mathcal{D}(A^{1/2})} \le C_R t^{-1/2} \|U_0 - V_0\|_X, \qquad 0 < t \le T_R,\ U_0,\ V_0 \in \mathcal{X}.$$

Therefore if we set $Z = \mathcal{D}(A^{1/2})$ and $t^* = T_R$ $(R = \sqrt{2C^*})$, then (6.2) is fulfilled. By Proposition 6.4, the condition (6.3) was already verified. Hence, by virtue of Theorem 6.1, we conclude the following main result.

THEOREM 6.2 *Let Ω be a two-dimensional bounded domain with Lipschitz boundary Γ and let Γ be split into two parts $\Gamma_D \neq \emptyset$ and Γ_N satisfying* (1.2) *and* (1.3)*, respectively. If* (1.4) *and* (1.5) *hold, then the dynamical system $(S(t), \mathcal{X}, X)$ determined from* (1.1) *possesses exponential attractors $\mathcal{M}$.*

We notice that the basin of attraction of $\mathcal{M}$ is the whole space $\mathcal{K}$ of initial data, for, in view of Proposition 6.1, $\mathcal{X}$ is an absorbing set of $(S(t), \mathcal{K}, X)$ and any bounded set of $\mathcal{K}$ is absorbed in $\mathcal{X}$ in finite time.

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7 Three-dimensional problem

In this section we shall consider the three-dimensional problem. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary Γ which is split into two parts $\Gamma_D \neq \emptyset$ and Γ_N satisfying (1.2) and (1.3).

As a matter of fact, we can argue in a similar way as for the two-dimensional problem. But in view of (2.8) we have to assume a stronger shift property that $\Lambda u \in L^2(\Omega)$ implies $u \in W^1_p(\Omega)$ with some $p > 3$ as well as the estimate

$$\|u\|_{W^1_p} \le C\|\Lambda u\|_{L^2}, \qquad u \in \Lambda^{-1}(L^2(\Omega)). \tag{7.1}$$

We do not know any general condition on Ω which ensures this shift property. We can only say that (7.1) is valid at least in some particular cases. For example, if Ω is a convex domain and if $\Gamma_D = \Gamma$ (i.e., $\Gamma_N = \emptyset$), then $\Lambda u \in L^2(\Omega)$ implies $u \in H^2(\Omega) \subset W^1_6(\Omega)$. Similarly, if Ω is a rectangular parallelepiped and if Γ_D and Γ_N consist of pairs of face-to-face side surfaces, then $\Lambda u \in L^2(\Omega)$ implies $u \in H^2(\Omega) \subset W^1_6(\Omega)$.

We set the same underlying space X defined by (3.1). In X, the initial and boundary value problem (1.1) is written as an abstract problem of the form

(3.2), where A and F are the same linear and nonlinear operators, respectively, as in the two-dimensional case. The only difference is that F has the domain

$$\mathcal{D}(F) = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \; u \in H^{3/p}(\Omega) \text{ and } v \in H^{3/p}(\Omega) \right\},$$

where $p > 3$ is the exponent appearing in (7.1). Then, $\mathcal{D}(A^\eta) \subset \mathcal{D}(F)$ with $\eta = (1/2) + (3/2p) < 1$ due to (2.5). Then F is shown to fulfill [22, (F)]. Therefore, by the same arguments used in Theorems 3.1 and 4.1, for any initial value $U_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$ in the space of initial values $\mathcal{K}$ given by (6.4), there exists a unique nonnegative local solution to (3.2) in the function space (3.9). In addition, by the *a priori* estimates provided by Proposition 5.1, the local solution is extended over the whole real semi-axis $[0, \infty)$.

Arguments in Section 6 are independent of the dimension. The set $\mathcal{B}$ given by (6.5) is an absorbing set and the set $\mathcal{X}$ given by (6.6) becomes a compact absorbing and invariant set. In this way, a dynamical system $(S(t), \mathcal{X}, X)$ is defined. This dynamical system has the global attractor $\mathcal{A} = \bigcap_{0 \le t < +\infty} S(t)\mathcal{X}$. Finally, we obtain the analogue of Theorem 6.2, that is,

THEOREM 7.1 *Let Ω be a three-dimensional bounded domain with Lipschitz boundary Γ and let Γ be split into two parts $\Gamma_D \neq \emptyset$ and Γ_N satisfying* (1.2) *and* (1.3)*, respectively. If* (1.4), (1.5) *and* (7.1) *hold, then the dynamical system $(S(t), \mathcal{X}, X)$ possesses exponential attractors $\mathcal{M}$.*

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Convergence to stationary states of solutions to the semilinear equation of viscoelasticity

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Abstract We consider the equation of viscoelasticity characterized by a nonlinear elastic force ϕ depending on the displacement u and subject to a time dependent external load. The dissipativity of the corresponding evolution system is only due to the presence of the relaxation kernel k. Rescaling $k(s)-k(\infty)$ with a relaxation time $\varepsilon > 0$, we can find a sufficiently small $\varepsilon_0 > 0$, such that, if ϕ is real analytic and $\varepsilon \in (0, \varepsilon_0]$, then any sufficiently smooth u converges to a single stationary state with an algebraic decay rate, provided that the external load suitably converges to a time independent one. The proof relies on the well-known Łojasiewicz-Simon inequality.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, be a bounded domain with smooth boundary $\partial\Omega$. We consider the following integrodifferential equation for a function $u : \Omega \times \mathbb{R} \to \mathbb{R}$

$$D_t^2 u(t) - \int_0^{+\infty} k(s)\Delta D_t u(t-s)ds + \phi(u(t)) = f(t), \qquad t > 0, \tag{1.1}$$

where $k : (0, +\infty) \to (0, +\infty)$ is a convex decreasing relaxation kernel such that $k(+\infty) > 0$, ϕ is a smooth function, and $f : \Omega \times (0, +\infty) \to \mathbb{R}$ is a source term. Given the past history $\tilde{u} : (-\infty, 0] \to \mathbb{R}$, we endow (1.1) with the following boundary and initial conditions

$$u(t)|_{\partial\Omega} = 0, \qquad \forall\, t \in \mathbb{R}, \tag{1.2}$$

$$u(t) = \tilde{u}(t), \qquad t \leq 0. \tag{1.3}$$

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Equation (1.1) is a model for a isothermal viscoelastic solid of Boltzmann type (e.g., a membrane). In this case, interpreting u as the displacement (see, e.g., [15, 36, 37]), the solid is subject to a nonlinear elastic force represented by ϕ and an external time dependent load f as well. Here we consider the scalar equation for the sake of simplicity only.

This kind of equation has been studied in [20] as a dynamical system on a suitable phase-space. Following the past history approach outlined in [21], the authors prove the existence of a global attractor assuming ϕ globally Lipschitz continuous. This result is far from being trivial since the memory term is the only source of dissipation for the associated dynamical system (compare, e.g., with [10, 12, 31, 33]) where (1.1) with an additional instantaneous damping $D_t u$ is analyzed). In [13], the understanding of the above problem has been greatly deepened. Indeed, for a time independent source, the authors show the existence of a global smooth attractor in the case of ϕ with critical (cubic) growth in three dimensions. Their argument is based on the existence of a global Lyapunov functional combined with appropriate uniform estimates. As the authors point out, thanks to their results, one can construct a family of exponential attractors which is stable (robust) with respect to a certain relaxation time $\varepsilon > 0$ related to a suitable rescaling of the relaxation kernel (see below, cf. also [11, 12]).

Here, within the setting of [13], we want to address to the problem of convergence of a single trajectory to a stationary state, i.e., a solution to the following elliptic problem

$$-k(+\infty)\Delta z + \phi(z) = f(+\infty), \qquad \text{in } \Omega,$$

$$z|_{\partial\Omega} = 0.$$

We recall that, in more than one dimension, the set of stationary solutions z can have a rather complicated structure and, for physically reasonable ϕ like, e.g., $\phi(r) = r^3 - ar$, $a > 0$, it can be a continuum (cf., for instance, [23, Rem.2.3.13]). Hence, for a dynamical system with the above stationary equation, it is not easy to establish sufficiently general criteria in order to decide whether a given trajectory does converge to a stationary solution. Indeed, this can even be false in some cases (see [5, 34, 35]). However, if ϕ is real analytic, a positive answer can be obtained by using the celebrated Łojasiewicz-Simon inequality (see [29, 30, 39], cf. also [27] for a simplified proof). Using this inequality, many people have been able to prove convergence results for several types of evolution equations or systems (see, for instance, [1, 2, 3, 4, 7, 8, 9, 18, 24, 26, 28, 38, 41, 42, 43]), even relaxing the analyticity assumption in some rather special cases (see [6, 25]). In particular, [7] is devoted to the analysis of a semilinear integrodifferential equation similar to (1.1) (see also [16, 17] for related problems), but characterized by the presence of a damping term of the form $BD_t u$, where B is a positive linear operator (for instance, $B = I$ or $B = -\Delta$) on $L^2(\Omega)$, and with no external time dependent load. The authors are able to demonstrate the convergence of a sufficiently smooth trajectory to a single stationary state under sufficiently general assumptions on the memory kernel. They also provide an estimate of the convergence rate to the steady state. However, they regard the precompactness of the trajectory as an assumption. Here, following [13], we first state conditions which ensure the precompactness of the trajectory in the phase space (see Theorem 2.1, cf. also Remark 3.2). This is not a trivial fact since ϕ can have critical growth if $N = 3$, so that Webb's compactness principle [40] does not apply (see also [24]). Then, we proceed to prove that results similar to [7] also hold for equation (1.1), provided that the kernel $k(s) - k(+\infty)$ is sufficiently close to the Dirac mass at 0. Of course, this restriction is not necessary if, like in [7], an extra damping term is added (see Remark 3.6 below). In order to do that, we shall follow the mentioned history space approach and the rescaling procedure devised in [11]. To handle the time dependent source, we shall use the ideas of [9] (see also [26]) combined with the further refinements contained in [22]. For the sake of simplicity, we shall consider only sources $f(t)$ which converge to 0 as t goes to ∞ (however, see Remark 3.5 below).

We now introduce the abstract formulation of problem (1.1)-(1.3). Set $H = L^2(\Omega)$ and denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the canonical inner product and the norm in H, respectively. Define the linear positive operator $A = -\Delta : \mathcal{D}(A) \subset H \to H$ with $\mathcal{D}(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and set, for any $r \in \mathbb{R}$,

$$V^r = \mathcal{D}(A^{(r+1)/2})$$

endowed with the inner product

$$\langle u_1, u_2\rangle_{V^r} = \langle A^{(r+1)/2}u_1, A^{(r+1)/2}u_2\rangle.$$

Identifying $H = V^{-1}$ with its dual space H^*, we have $(V^r)^* = V^{-r-2}$. Also, we recall that $V^{r_1} \hookrightarrow V^{r_2}$ with compact injection for any r_1, r_2 such that $r_1 > r_2$.

Let $\phi \in C^2(\mathbb{R})$ such that

$$\phi(0) = 0, \tag{1.4}$$

$$|\phi''(y)| \le c(1 + |y|), \qquad \forall\, y \in \mathbb{R}, \tag{1.5}$$

$$\liminf_{|y|\to+\infty} \frac{\phi(y)}{y} > -k(+\infty)\lambda_1, \tag{1.6}$$

where λ_1 is the first eigenvalue of A. Then, equations (1.1) and (1.2) can be formulated as a single abstract integrodifferential evolution equation, that is,

$$D_t^2 u(t) + k(+\infty)Au(t) + \int_0^{+\infty} (k(s) - k(+\infty))AD_t u(t-s)ds + \phi(u(t)) = f(t), \quad t > 0. \tag{1.7}$$

This abstract equation is endowed with the initial condition (1.3). Notice that we have implicitly assumed that $A\tilde{u}(-\infty) = 0$.

2 Preliminary results

Let us set

$$\mu(s) = -k'(s).$$

To avoid the presence of unnecessary constants we can assume $k(0) = 2$ and $k(+\infty) = 1$ without any loss of generality. Then we suppose that

$$\mu \in W^{1,1}(\mathbb{R}^+), \tag{2.1}$$

$$\mu(s) \geq 0, \quad \forall\, s \in \mathbb{R}^+, \tag{2.2}$$

$$\mu'(s) + \delta\mu(s) \leq 0, \qquad \text{for a.e. } s \in \mathbb{R}^+, \tag{2.3}$$

for some $\delta > 0$. Note that (2.1) implies

$$\lim_{s \to 0} \mu(s) = \mu_0 < \infty. \tag{2.4}$$

Furthermore, for any relaxation time $\varepsilon \in (0, 1]$, following [11] (see also [12]), we set

$$\mu_\varepsilon(s) = \frac{1}{\varepsilon^2}\mu\Big(\frac{s}{\varepsilon}\Big).$$

We notice that

$$\int_0^{+\infty} \mu(s)ds = 1,$$

which entails

$$\int_0^{+\infty} \mu_\varepsilon(s)ds = \frac{1}{\varepsilon} \qquad \text{and} \qquad \int_0^{+\infty} s\mu_\varepsilon(s)ds = 1. \tag{2.5}$$

Then, we introduce the weighted Hilbert spaces $\mathcal{M}_\varepsilon^r = L^2_{\mu_\varepsilon}(\mathbb{R}^+; V^{r-1})$, endowed with the inner products

$$\langle \eta_1, \eta_2 \rangle_{\mathcal{M}_\varepsilon^r} = \int_0^{+\infty} \mu_\varepsilon(s) \langle \eta_1(s), \eta_2(s) \rangle_{V^{r-1}} ds.$$

We shall make use of the infinitesimal generator of the right-translation semigroup on $\mathcal{M}_\varepsilon^1$, namely, the linear operator $T_\varepsilon = -\partial_s$ (∂_s being the distributional derivative with respect to s) with domain

$$\mathcal{D}(T_\varepsilon) = \big\{\eta \in \mathcal{M}_\varepsilon^1 \,:\, \partial_s \eta \in \mathcal{M}_\varepsilon^1,\ \eta(0) = 0\big\}.$$

On account of (2.3), there holds

$$\langle T_\varepsilon \eta, \eta \rangle_{\mathcal{M}^0_\varepsilon} \leq -\frac{\delta}{2\varepsilon} \|\eta\|^2_{\mathcal{M}^0_\varepsilon}, \qquad \forall \eta \in \mathcal{D}(T_\varepsilon). \tag{2.6}$$

Setting now

$$\eta^t(s) = u(t) - u(t-s),$$

equation (1.7) can be rewritten as the evolution system

$$\begin{cases} D_t^2 u + \mathcal{G}(u) + \displaystyle\int_0^{+\infty} \mu_\varepsilon(s) A \eta(s) ds = f(t), \\ D_t \eta = T_\varepsilon \eta + D_t u, \end{cases} \tag{2.7}$$

which is subject to the initial conditions

$$u(0) = u_0, \quad D_t u(0) = u_1, \quad \eta^0 = \eta_0,$$

where (cf. (1.3))

$$u_0 = \tilde{u}(0), \quad u_1 = D_t \tilde{u}(0), \quad \eta_0(s) = \tilde{u}(0) - \tilde{u}(-s).$$

Here the nonlinear operator $\mathcal{G} : V^1 \to H$ is defined by

$$\mathcal{G}(u) = Au + \phi(u).$$

Note that the formal limit of system (2.7) as ε goes to 0 is the well-known strongly damped semilinear wave equation (see [32] and references therein).

We now introduce the Banach spaces

$$\mathcal{V}^r_\varepsilon = V^r \times V^{r-1} \times \mathcal{M}^{r+1}_\varepsilon, \quad \mathcal{W}^r_\varepsilon = V^r \times V^{r-1} \times \mathcal{L}^{r+1}_\varepsilon,$$

where

$$\mathcal{L}^r_\varepsilon = \mathcal{M}^r_\varepsilon \cap \mathcal{D}(T_\varepsilon)$$

is equipped with the norm

$$\|\eta\|^2_{\mathcal{L}^r_\varepsilon} = \|\eta\|^2_{\mathcal{M}^r_\varepsilon} + \varepsilon \|T_\varepsilon \eta\|^2_{\mathcal{M}^{r-1}_\varepsilon} + \varepsilon \sup_{x \geq 1} x \int_{(0,1/x)\cup(x,+\infty)} \mu_\varepsilon(s) \|\eta(s)\|^2_{V^{r-2}} ds.$$

Then we recall (see [19, Lemma 5.1], cf. also [33, Lemma 5.5]) that the embedding $\mathcal{L}^{r+1}_\varepsilon \subset \mathcal{M}^r_\varepsilon$ is compact. There holds

THEOREM 2.1 *Let* (1.4)–(1.6), (2.1)–(2.3) *hold. If*

$$f \in W^{1,\infty}(\mathbb{R}^+, H), \tag{2.8}$$

then (2.7) *defines a process* $U_\varepsilon(t, \tau)$ *on* $\mathcal{V}^0_\varepsilon$ *and, for any* $(u_0, u_1, \eta_0) \in \mathcal{W}^1_\varepsilon$, *setting*

$$(u(t), D_t u(t), \eta^t) = U_\varepsilon(t, 0)(u_0, u_1, \eta_0),$$

we have that $\bigcup_{t\geq 0}(u(t), D_t u(t), \eta^t)$ *is bounded in* $\mathcal{W}^1_\varepsilon$ *and precompact in* $\mathcal{V}^0_\varepsilon$.

The proof can be obtained from previous results proved in the case $f \equiv 0$. In fact, for the existence of the process one can argue as in [12] (see also [33]). On the other hand, adapting [13, Lemma 5.3] we can infer that the trajectory $\bigcup_{t\geq 0}(u(t), D_t u(t), \eta^t)$ is bounded in $\mathcal{V}^1_\varepsilon$. Moreover, the boundedness of the second component in $\mathcal{L}^2_\varepsilon$ comes again from [11, Lemmata 3.3 and 3.4]. The precompactness then follows from $\mathcal{L}^2_\varepsilon \Subset \mathcal{M}^1_\varepsilon$.

Let us define the set

$$\mathcal{S} = \{v_\infty \in V^1 \,:\, \mathcal{G}(v_\infty) = 0\},$$

and introduce the energy functional

$$E(v) = \frac{1}{2}\|v\|^2_{V^0} + \langle \Phi(v), 1\rangle, \qquad \forall\, v \in V^0,$$

where $\Phi(y) = \int_0^y \phi(\xi)d\xi$. Observe that, due to the assumptions (1.4)–(1.6), the set $\mathcal{S}$ is bounded in V^1 and, consequently, in $L^\infty(\Omega)$.

We can now state and prove a quite standard result of convergence.

LEMMA 2.1 *Let* (1.4)–(1.6), (2.1)–(2.3), *and* (2.8) *hold. Moreover, suppose that*

$$\int_0^{+\infty} \|f(\tau)\|_{V^{-2}}\, d\tau < +\infty. \tag{2.9}$$

Consider $(u_0, u_1, \eta_0) \in \mathcal{W}^1_\varepsilon$ *and set* $(u(t), D_t u(t), \eta^t) = U_\varepsilon(t,0)(u_0, u_1, \eta_0)$. *Then, we have*

$$\frac{1}{\varepsilon}\int_0^{+\infty} \|\eta^t\|^2_{\mathcal{M}^1_\varepsilon}\, dt \leq C_0, \tag{2.10}$$

for some positive constant C_0 *independent of* ε. *In addition, there hold*

$$\lim_{t\to+\infty} \eta^t = 0, \qquad \textit{strongly in } \mathcal{M}^1_\varepsilon, \tag{2.11}$$

$$\lim_{t\to+\infty} D_t u(t) = 0, \qquad \textit{strongly in } V^{-1}, \tag{2.12}$$

$$\omega(u_0, u_1, \eta_0) \subseteq \{(w^1, w^2, w^3) \in \mathcal{W}^1_\varepsilon \,:\, w^1 \in \mathcal{S},\ w^2 \equiv 0,\ w^3 \equiv 0\}, \tag{2.13}$$

and E *is constant on the set* $\{u_\infty \in V^1 \,:\, (u_\infty, 0, 0) \in \omega(u_0, u_1, \eta_0)\}$.

Proof. Following [13], we define the functional

$$\Lambda(t) = \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}^1_\varepsilon} + \frac{1}{2}\|D_t u(t)\|^2 + E(u(t)) + \int_t^{+\infty} \langle f(\tau), D_t u(\tau)\rangle d\tau,$$

and observe that

$$D_t\Lambda(t) - \frac{1}{2}\int_0^{+\infty} \mu'_\varepsilon(s)\|A^{1/2}\eta^t(s)\|^2\, ds = 0.$$

Thus, on account of (2.3),

$$D_t\Lambda(t) \le -\frac{\delta}{2\varepsilon}\|\eta^t\|^2_{\mathcal{M}^1_\varepsilon}.$$

Hence, owing to Theorem 2.1 and (2.9), Λ is decreasing and bounded. This entails the bound (2.10). Set now

$$h(t) = \|\eta^t\|^2_{\mathcal{M}^0_\varepsilon}, \qquad \forall\, t \ge 0,$$

and notice that, using the second equation of (2.7),

$$h'(t) = 2\langle \eta^t, \eta^t_t\rangle_{\mathcal{M}^0_\varepsilon} = 2\langle \eta^t, T_\varepsilon\eta + D_t u\rangle_{\mathcal{M}^0_\varepsilon}.$$

Hence h' is globally bounded. Thus, on account of (2.10),

$$\lim_{t\to+\infty} \eta^t = 0, \qquad \text{strongly in } \mathcal{M}^0_\varepsilon,$$

and the precompactness of the trajectory implies (2.11). On the other hand, from the second equation of system (2.7), using again the precompactness of the trajectory, we also deduce (2.12). Consequently, any point of $\omega(u_0, u_1, \eta_0)$ is of the form $(u_\infty, 0, 0)$. Let $\{t_n\}_{n\in\mathbb{N}}$ be an unbounded increasing sequence such that $u(t_n) \to u_\infty$ in V^0, as n goes to ∞. Therefore $\mathcal{G}(u(t_n+\tau)) \to \mathcal{G}(u_\infty)$ in V^{-2}, as n tends to ∞, for any $\tau \in (0,1]$. Then, using the first equation of system (2.7) and denoting by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality pairing between V^{-2} and V^0, we have, on account of (2.11) and (2.12),

$$\begin{aligned}
&\langle\langle \mathcal{G}(u_\infty), v\rangle\rangle \\
&= \langle\langle \int_0^1 \mathcal{G}(u_\infty), v\rangle\rangle d\tau \\
&= \lim_{n\to+\infty} \int_0^1 \langle\langle \mathcal{G}(u(t_n+\tau)), v\rangle\rangle d\tau \\
&= \lim_{n\to+\infty} \int_0^1 \langle\langle -u_{tt}(t_n+\tau) - \int_0^{+\infty} \mu_\varepsilon(s) A\eta^{t_n+\tau}(s)ds + f(t_n+\tau), v\rangle\rangle d\tau \\
&= 0,
\end{aligned}$$

for any $v \in V^0$. Thus $\mathcal{G}(u_\infty) = 0$ so that (2.13) holds. Finally, it is clear that $\lim_{t\to+\infty}\Lambda(t) = \Lambda_\infty$, where $E(u_\infty) = \Lambda_\infty$ for all $u_\infty \in V^1$ such that $(u_\infty, 0, 0) \in \omega(u_0, u_1, \eta_0)$.

3 Main result

We first report the main tool of this section, namely, the Łojasiewicz-Simon inequality in the following form (see [24, Thm. 2.2 and Prop. 5.3.1].

LEMMA 3.1 *Suppose that ϕ is real analytic and assume* (1.4)–(1.6). *Let $v_\infty \in \mathcal{S}$. Then there exist $\rho \in (0, \frac{1}{2})$, $\sigma > 0$, and $C_1 > 0$ such that*

$$\|\mathcal{G}(v)\|_{V^{-2}} \geq C_1 |E(v) - E(v_\infty)|^{1-\rho}, \tag{3.1}$$

for all $v \in V^0$ such that $\|v - v_\infty\|_{V^0} < \sigma$.

REMARK 3.1 If $\rho_0 < \rho$, then one can always find $\sigma_0 \leq \sigma$ such that inequality (3.1) holds with ρ and σ replaced by ρ_0 and σ_0, respectively.

Our main result is

THEOREM 3.1 *Let* (2.1)–(2.3), (2.8), *and* (2.9) *hold. In addition, suppose*

$$\sup_{t \geq 0} t^{1+\theta} \int_t^{+\infty} \|f(\tau)\|^2 \, d\tau < +\infty, \tag{3.2}$$

for some $\theta > 0$, and assume that ϕ is real analytic and satisfies (1.4)–(1.6). *Take $(u_0, u_1, \eta_0) \in \mathcal{W}^1_\varepsilon$ and set $(u(t), D_t u(t), \eta^t) = U_\varepsilon(t, 0)(u_0, u_1, \eta_0)$, for all $t \geq 0$. Then, there exists $\varepsilon_0 > 0$ such that, for any fixed $\varepsilon \in (0, \varepsilon_0]$, $\omega(u_0, u_1, \eta_0)$ consists of a single point $(u_\infty, 0, 0)$ and, as t goes to $+\infty$,*

$$u(t) \to u_\infty, \qquad \text{strongly in } V^0. \tag{3.3}$$

If

$$\theta > \frac{2\rho}{1 - 2\rho}, \tag{3.4}$$

then one can find $t^* > 0$ and a positive constant C_2 such that

$$\|u(t) - u_\infty\| \leq C_2 \, t^{-\rho/(1-2\rho)}, \qquad \forall t \geq t^*. \tag{3.5}$$

Otherwise, one can find $\rho_0 \in (0, \rho)$ so that

$$\theta > \frac{2\rho_0}{1 - 2\rho_0}, \tag{3.6}$$

a time $t^{**} > 0$ and a positive constant C_3 such that

$$\|u(t) - u_\infty\| \leq C_3 \, t^{-\rho_0/(1-2\rho_0)}, \qquad \forall t \geq t^{**}. \tag{3.7}$$

REMARK 3.2 Consider for simplicity the homogeneous case $f \equiv 0$. If $(u_0, u_1, \eta_0) \in \mathcal{V}^0_\varepsilon$, then the corresponding trajectory $\mathbf{z}(t) = (u(t), D_t u(t), \eta^t)$ can be written as $\mathbf{z} = \mathbf{z}_d + \mathbf{z}_c$, where $\mathbf{z}_d(t)$ exponentially decays to 0 as t goes to $+\infty$, while $\mathbf{z}_c(t)$ is bounded in $\mathcal{W}^{1/4}_\varepsilon$ for any $t \geq 0$ (cf. [13, Secs. 6 and 7]). The trajectory is thus precompact in $\mathcal{V}^0_\varepsilon$ and it can still be proved that

$\omega(u_0, u_1, \eta_0) = \{(u_\infty, 0, 0)\}$, and (3.3) as well as (3.5) hold, whenever $\varepsilon > 0$ is small enough.

REMARK 3.3 Since $\mathcal{S}$ is a bounded subset of $L^\infty(\Omega)$, the assumption on the analyticity of ϕ can be slightly relaxed. Indeed, we can suppose that ϕ is real analytic on a suitable bounded interval $[-M, M]$ with $M > 0$ such that $\sup_{v_\infty \in \mathcal{S}} \|v_\infty\|_{L^\infty(\Omega)} < M$. In this case, however, instead of Lemma 3.1 we have to use a localized version of Łojasiewicz-Simon inequality (see [2, 18]).

REMARK 3.4 Notice that, by interpolation, we can get estimates similar to (3.5) and (3.7) for $\|u(t) - u_\infty\|_{V^0}$.

REMARK 3.5 Lemma 3.1 also holds when the nonlinear function ϕ depends on x (see [24]). In particular, this allows to consider a source term F which converges to some $F_\infty \in H$ as t goes to $+\infty$. Our results still hold provided $\mathcal{G}(u)$ and f are defined as follows

$$\mathcal{G}(u) = Au + \phi(u) - F_\infty, \qquad f = F - F_\infty.$$

In this case, the energy functional is

$$E(v) = \frac{1}{2}\|v\|^2_{V^0} + \langle \Phi(v), 1\rangle - \langle F_\infty, v\rangle.$$

REMARK 3.6 If equation (1.7) contains a damping term like $\kappa D_t u$, then Theorem 3.1 holds without any restriction on ε. Moreover, assumption (2.9) in Lemma 2.1 can be replaced by (2.8). Consequently, in Theorem 3.1, we only need (3.2).

REMARK 3.7 The above results and remarks as well as the main theorem stated in the next section still hold for the case of a viscoelastic Kirchhoff plate subject to a nonlinear force depending on the vertical deflection u and subject to an external time dependent load. More precisely, for $u : \Omega \times \mathbb{R} \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ being a bounded domain with smooth boundary $\partial\Omega$, we consider the integrodifferential evolution equation

$$D_t^2 u(t) + k(+\infty)\Delta^2 u(t) + \int_0^{+\infty} (k(s) - k(+\infty))\Delta^2 D_t u(t-s)ds + \phi(u(t)) = f(t),$$

endowed with the Navier boundary conditions

$$u = \Delta u = 0, \qquad \text{on } \partial\Omega \times \mathbb{R}.$$

Let us define $H = L^2(\Omega)$ and introduce the linear positive operator $A = \Delta^2 : \mathcal{D}(A) \subset H \to H$ with

$$\mathcal{D}(A) = \big\{z \in H^4(\Omega) : z = \Delta z = 0, \ \text{ on } \partial\Omega\big\}.$$

Arguing as above, we can write the corresponding dynamical system that has exactly the form (2.7). Notice that, in the present case, the growth condition (1.5) on ϕ can be neglected since the Lyapunov functional implies that u is globally bounded in $\Omega \times \mathbb{R}^+$. In fact, here we have $V^0 = D(A^{1/2}) = H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow C^0(\bar{\Omega})$. Thus, in particular, owing to Remark 3.4, an estimate of the convergence rate in the C^0-norm holds. Similar considerations are valid for the semilinear Bernoulli viscoelastic beam (compare with [7, 16, 17]).

4 Proof of Theorem 3.1

Let $(u_\infty, 0, 0) \in \omega(u_0, u_1, \eta_0)$. In this proof, we will denote with c a generic positive constant, independent of ε, which may vary even in the same line.

In order to prove (3.3) we proceed along the lines of [9, Proof of Thm.2.3]. First let us assume (3.4) and recall that θ comes from (3.2), while ρ comes from (3.1). Then, introduce the unbounded set

$$\Sigma = \Big\{ t \geq 0 \,:\, \|u(t) - u_\infty\|_{V^0} \leq \frac{\sigma}{3} \Big\},$$

where σ is given by Lemma 3.1. For every $t \in \Sigma$, define

$$\tau(t) = \sup \Big\{ t' \geq t \,:\, \sup_{s \in [t,t']} \|u(s) - u_\infty\|_{V^0} \leq \sigma \Big\}$$

and observe that $\tau(t) > t$, for every $t \in \Sigma$.

Let $t_0 \in \Sigma$ be large enough such that

$$\|D_t u(t)\| + \|\eta^t\|_{\mathcal{M}^1_\varepsilon} \leq 1, \qquad \forall\, t \geq t_0,$$

and set

$$\begin{aligned}
J &= [t_0, \tau(t_0)),\\
J_1 &= \Big\{ t \in J \,:\, \mathcal{N}(u, D_t u, \eta)(t) > \Big(\int_t^{+\infty} \|f(s)\|^2 \, ds \Big)^{1-\rho} \Big\},\\
J_2 &= J \setminus J_1,\\
J_3 &= \{ t \geq 0 \,:\, \beta \mathcal{N}(u, D_t u, \eta)(t) \leq \|f(t)\| \},
\end{aligned}$$

where $\beta > 0$ is to be fixed below and

$$\mathcal{N}(u, D_t u, \eta)(t) = \|\mathcal{G}(u(t))\|_{V^{-2}} + \|D_t u(t)\| + \frac{1}{\sqrt{\varepsilon}} \|\eta^t\|_{\mathcal{M}^1_\varepsilon}.$$

We now introduce the functional

$$\begin{aligned}\Lambda_0(t) = \frac{1}{2}\|\eta^t\|^2_{\mathcal{M}^1_\varepsilon} + \frac{1}{2}\|D_t u(t)\|^2 + E(u(t)) - E(u_\infty) \\ + \int_t^{\tau(t_0)} \langle f(s), D_t u(s)\rangle \chi(s)\, ds \\ +\alpha\varepsilon \int_t^{\tau(t_0)} \left(\langle f(s), A^{-1}\mathcal{G}(u(s))\rangle - \langle f(s), \eta^s\rangle_{\mathcal{M}^0_\varepsilon}\right) \chi(s)\, ds \\ +\alpha\varepsilon \left(\langle D_t u(t), A^{-1}\mathcal{G}(u(t))\rangle - \langle D_t u(t), \eta^t\rangle_{\mathcal{M}^0_\varepsilon}\rangle\right),\end{aligned}$$

for $\alpha \in (0,1)$ and every $t \in J$, where χ is the characteristic function of J_3.

On account of (2.7), we have

$$\begin{aligned}D_t\Lambda_0 = \frac{1}{2}\int_0^{+\infty} \mu'_\varepsilon(s)\|A^{1/2}\eta^t(s)\|^2 ds \\ -\alpha\|D_t u\|^2 - \alpha\varepsilon\|\mathcal{G}(u)\|^2_{V^{-2}} + \alpha\varepsilon\langle D_t u, [A^{-1}\mathcal{G}(u)]_t\rangle \\ +\langle f, D_t u\rangle(1-\chi) + \alpha\varepsilon\left(\langle f, A^{-1}\mathcal{G}(u)\rangle - \langle f, \eta\rangle_{\mathcal{M}^0_\varepsilon}\right)(1-\chi) \\ +\alpha\varepsilon\left\| \int_0^{+\infty} \mu_\varepsilon(s) A^{1/2}\eta(s) ds\right\|^2 - \alpha\varepsilon\int_0^{+\infty} \mu'_\varepsilon(s)\langle \eta(s), D_t u\rangle\, ds.\end{aligned}$$

Observe that

$$\langle D_t u, [A^{-1}\mathcal{G}(u)]_t\rangle = \|D_t u\|^2 + \langle D_t u, [A^{-1}\phi(u)]_t\rangle. \tag{4.1}$$

Then, thanks to (1.5) and (2.5), we obtain

$$\langle D_t u, [A^{-1}\phi(u)]_t\rangle \leq c\|D_t u\|^2,$$

$$\alpha\varepsilon\left\| \int_0^{+\infty} \mu_\varepsilon(s) A^{1/2}\eta(s) ds\right\|^2 \leq \alpha\|\eta\|^2_{\mathcal{M}^1_\varepsilon},$$

$$-\alpha\varepsilon\int_0^{+\infty} \mu'_\varepsilon(s)\langle \eta(s), D_t u\rangle\, ds \leq c\alpha^2\|D_t u\|^2 - \frac{1}{4}\int_0^{+\infty} \mu'_\varepsilon(s)\|A^{1/2}\eta(s)\|^2\, ds.$$

Moreover, we easily infer

$$\begin{aligned}\langle f, D_t u\rangle(1-\chi) \leq \beta\mathcal{N}(u, D_t u, \eta)^2, \\ \alpha\varepsilon\langle f, A^{-1}\mathcal{G}(u)\rangle(1-\chi) \leq \beta c\mathcal{N}(u, D_t u, \eta)^2, \\ \alpha\varepsilon\langle f, \eta\rangle_{\mathcal{M}^0_\varepsilon}(1-\chi) \leq \alpha\beta c\mathcal{N}(u, D_t u, \eta)^2.\end{aligned}$$

Thus we deduce

$$\begin{aligned}D_t\Lambda_0 \leq -\left(\frac{\delta}{4\varepsilon} - \alpha\right)\|\eta^t\|^2_{\mathcal{M}^1_\varepsilon} - \alpha[(1+\varepsilon) - c(\varepsilon+\alpha)]\|D_t u\|^2 - \alpha\varepsilon\|\mathcal{G}(u)\|^2_{V^{-2}} \\ +\beta[1 + c(\alpha+1)]\mathcal{N}(u, D_t u, \eta)^2.\end{aligned}$$

Choosing $\alpha \in (0,1)$ and β sufficiently small, we can find $\varepsilon_0 > 0$ and a positive constant $\gamma_0 = \gamma_0(\alpha, \beta, \varepsilon_0) > 0$ such that

$$D_t \Lambda_0 \leq -\gamma_0 \mathcal{N}(u, D_t u, \eta)^2, \qquad \forall\, t \geq 0, \tag{4.2}$$

for any $\varepsilon \in (0, \varepsilon_0]$, which henceforth is assumed to be fixed. Inequality (4.2) implies that Λ_0 is decreasing. Therefore, since

$$D_t\big(|\Lambda_0(t)|^\rho \operatorname{sgn} \Lambda_0(t)\big) = \rho |\Lambda_0(t)|^{\rho-1} D_t \Lambda_0(t), \qquad t \in J, \tag{4.3}$$

the function $|\Lambda_0|^\rho \operatorname{sgn} \Lambda_0$ is decreasing as well.

Using (3.1), for every $t \in J_1$, we have

$$|\Lambda_0(t)|^{1-\rho} \leq c \mathcal{N}(u, D_t u, \eta)(t).$$

Consequently, we infer

$$\begin{aligned}\int_{J_1} \mathcal{N}(u, D_t u, \eta)(s)\, ds &\leq -c \int_{t_0}^{\tau(t_0)} D_t\big(|\Lambda_0(s)|^\rho \operatorname{sgn} \Lambda_0(s)\big) ds \\ &\leq c\Big(|\Lambda_0(t_0)|^\rho + |\Lambda_0(\tau(t_0))|^\rho\Big),\end{aligned}$$

where $|\Lambda_0(\tau(t_0))| = 0$ if $\tau(t_0) = +\infty$.

On the other hand, if $t \in J_2$, by definition of J_2 and (3.2), we deduce

$$\mathcal{N}(u, D_t u, \eta)(t) \leq \Big(\int_t^{+\infty} \|f(s)\|^2\, ds\Big)^{1-\rho} \leq c t^{-(1+\theta)(1-\rho)}. \tag{4.4}$$

Hence, on account of (3.4), we can integrate $\mathcal{N}(u, D_t u, \eta)$ over J_2 to get

$$\int_{J_2} \mathcal{N}(u, D_t u, \eta)(s)\, ds \leq c t_0^{-\theta+\rho+\rho\theta}.$$

Thus, in particular, $\|D_t u\|$ is integrable over J and, due to Lemma 2.1 and (3.2),

$$\begin{aligned}0 &\leq \limsup_{t_0 \in \Sigma,\, t_0 \to +\infty} \int_{t_0}^{\tau(t_0)} \|D_t u(s)\|\, ds \\ &\leq c \limsup_{t_0 \in \Sigma,\, t_0 \to +\infty} \Big(|\Lambda_0(t_0)|^\rho + |\Lambda_0(\tau(t_0))|^\rho + t_0^{-\theta+\rho+\rho\theta}\Big) = 0.\end{aligned} \tag{4.5}$$

Notice that, for every $t \in J$,

$$\|u(t) - u_\infty\| \leq \int_{t_0}^t \|D_t u(s)\| ds + \|u(t_0) - u_\infty\|. \tag{4.6}$$

Suppose now that $\tau(t_0) < +\infty$ for any $t_0 \in \Sigma$. By definition, we have

$$\|u(\tau(t_0)) - u_\infty\|_{V^1} = \sigma, \qquad \forall\, t_0 \in \Sigma.$$

Consider an unbounded sequence $\{t_n\}_{n\in\mathbb{N}} \subset \Sigma$ such that

$$\lim_{n\to+\infty} \|u(t_n) - u_\infty\|_{V^1} = 0.$$

By compactness, we can find a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and an element $\tilde{u}_\infty \in \mathcal{S}$ such that $\|\tilde{u}_\infty - u_\infty\|_{V^1} = \sigma$ and

$$\lim_{k\to+\infty} \|u(\tau(t_{n_k})) - \tilde{u}_\infty\|_{V^1} = 0.$$

Then, owing to (4.5) and (4.6), we deduce the contradiction

$$0 < \|\tilde{u}_\infty - u_\infty\| \leq \limsup_{k\to+\infty} \Big(\int_{t_{n_k}}^{\tau(t_{n_k})} \|D_t u(\tau)\| \, d\tau + \|u(t_{n_k}) - u_\infty\| \Big) = 0,$$

so that $\tau(t_0) = +\infty$ for some $t_0 > 0$ large enough. We can thus conclude that $\|D_t u(\cdot)\|$ is indeed integrable over (t_0, ∞) so that, by compactness, (3.3) follows. Finally, to obtain (3.5) and (3.7), we follow exactly the argument devised in [22].

REMARK 4.1 In the linear case, that is, $\phi \equiv 0$, we have $\mathcal{G}(u) = Au$. Thus, recalling (4.1), we now have $\langle D_t u, D_t[A^{-1}\mathcal{G}(u)]\rangle = \|D_t u\|^2$, and, if $f \equiv 0$, it is possible to recover from (4.2) the well-known exponential decay of the solution (cfr. [14]). On the other hand, as a byproduct, here we have also a decay estimate in the presence of a time dependent body force which suitably converges to a time independent one (see Remark 3.5).

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Asymptotic behavior of a phase-field system with dynamic boundary conditions

Stefania Gatti and Alain Miranville

Abstract This article is devoted to the study of the asymptotic behavior of a Caginalp phase-field system with nonlinear dynamic boundary conditions. As a proper parameter ε goes to zero, this problem converges to the viscous Cahn-Hilliard equation. We first prove the existence and uniqueness of the solution to the system and then provide an upper semicontinuous family of global attractors $\{\mathcal{A}_\varepsilon\}$. Furthermore, we prove the existence of an exponential attractor for each problem, which yields, since it contains the aforementioned global attractor, the finite fractal dimensionality of $\mathcal{A}_\varepsilon$.

1 Introduction

We are concerned in this article with the well-posedness and the longtime behavior of a one-parameter family of phase-field type equations with nonlinear dynamic (in the sense that the time derivative of the unknowns also appear) boundary conditions. These have recently been proposed by physicists (see [4], [5], [7] and references therein) to model phase separations in confined systems, for which the interactions with the walls need to be taken into account.

To be more precise, we consider a two-phase Caginalp type system, whose state is described by the temperature w and the phase-field (or order parameter) u, occupying a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\Gamma = \partial\Omega$. For each parameter $\varepsilon \in [0,1]$, we will deal with the following problem:

$$\begin{cases} \varepsilon w_t - \Delta w = -u_t, \qquad t>0, \quad x \in \Omega \\ u_t - \Delta u + f(u) = w, \qquad t>0, \quad x\in\Omega \\ \partial_{\boldsymbol{n}} w|_{\partial\Omega} = 0, \qquad t>0, \quad x\in\Gamma \\ u_t - \Delta_\Gamma u + \lambda u + \partial_{\boldsymbol{n}} u + g(u) = 0, \qquad t>0, \quad x\in\Gamma \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \qquad x\in\Omega, \end{cases}$$

where Δ_Γ is the Laplace-Beltrami operator, $\partial_{\boldsymbol{n}}$ is the outward normal derivative and $\lambda > 0$. Notice that, when $\varepsilon = 0$ (we will refer to this situation as the limiting case), the system is equivalent to the viscous Cahn-Hilliard equation proposed in [11], supplemented with dynamic boundary conditions. In order to study the problem, it is convenient to introduce, following [10], a further variable $\psi = u|_\Gamma$ and to view the dynamic boundary conditions as a parabolic equation for ψ on the boundary, namely,

$$\psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}} u + g(\psi) = 0, \qquad t > 0, \quad x \in \Gamma$$
$$u|_\Gamma = \psi, \qquad t > 0, \quad x \in \Gamma$$
$$\psi|_{t=0} = \psi_0, \qquad x \in \Gamma,$$

while the other equations remain unchanged.

This model, without the nonlinearity g and with stronger assumptions on f, has been considered in [1] (see also, e.g., [9] for the case of classical boundary conditions), where the existence of solutions in H^1-spaces is proven, together with the convergence of the solutions to steady states. Here, the weaker assumptions on the nonlinearities force the choice of more regular phase-spaces (namely, H^2 instead of H^1). The approach that we adopt has been developed in several recent articles on the Cahn-Hilliard equation with dynamic boundary conditions [1, 10, 12, 13, 16]. Following in particular [10], we obtain several *a priori* estimates which, employing L^p-techniques, as well as the Leray-Schauder fixed point theorem [17], furnish the existence of a solution. Besides, relying again on the *a priori* estimates, we prove the Lipschitz continuous dependence of the solutions on the initial data at any fixed time. These preliminary results show that the problem generates a dissipative dynamical system in a proper phase-space. Next, by suitably decomposing the semigroup, we obtain the existence of smooth global attractors $\mathcal{A}_\varepsilon$. We recall that the global attractor is the unique compact and invariant set which attracts the bounded sets of initial data as time goes to $+\infty$. Actually, applying the procedure devised in [6], we see that the family $\{\mathcal{A}_\varepsilon\}$ is upper semicontinuous at zero. Furthermore, for every $\varepsilon \in [0,1]$, we can construct, thanks to an abstract result from [2], an exponential attractor $\mathcal{M}_\varepsilon$, which is a compact and positively invariant set which has final fractal dimension and attracts exponentially fast the bounded sets of initial data. Unfortunately, we are not able to construct a robust (i.e., continuous) family of exponential attractors. Nevertheless, the nondependence of proper constants on ε furnishes a uniform bound on the fractal dimension of $\mathcal{M}_\varepsilon$. Moreover, noting that the global attractor $\mathcal{A}_\varepsilon$ is contained in $\mathcal{M}_\varepsilon$, it follows that the fractal dimension of $\mathcal{A}_\varepsilon$ is uniformly bounded with respect to ε as well.

In order to properly define the phase-spaces, which will depend on the parameter ε, but will always have three components, we need an equation for w in the limiting case $\varepsilon = 0$. For this purpose, we introduce the operator $A = \mathbb{I} - \Delta : \mathcal{D}(A) \to L^2(\Omega)$, where $\mathcal{D}(A) = \{w \in H^2(\Omega) : \quad \partial_{\boldsymbol{n}} w|_{\partial\Omega} = 0\}$. Then, from the second equation of our system with $\varepsilon = 0$, we have $w = \mathcal{J}(u)$, where $\mathcal{J} : H^2(\Omega) \to \mathcal{D}(A)$ is defined by

$$\mathcal{J}(u) = A^{-1}\big(- \Delta u + f(u)\big),$$

and, as in [9], $\mathcal{J} \in C^1(H^2(\Omega); \mathcal{D}(A))$. Next, setting $\langle u \rangle = \frac{1}{|\Omega|}\int_\Omega u dx$, we observe that the solutions verify the conservation law

$$\langle u(t)\rangle + \varepsilon \langle w(t)\rangle = I_\varepsilon, \qquad \forall t \geq 0. \tag{1.1}$$

We now introduce the phase-spaces, which take into account (1.1), by

$$\mathbb{D}^\varepsilon_M = \{(w, u, \psi) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Gamma) : \partial_{\boldsymbol{n}} w|_{\partial\Omega} = 0,\ u|_{\partial\Omega} = \psi,\ |I_\varepsilon| \leq M\},$$

when $\varepsilon > 0$, while

$$\mathbb{D}^0_M = \{(w, u, \psi) \in H^2(\Omega) \times H^2(\Omega) \times H^2(\Gamma) : w = \mathcal{J}(u),\ u|_{\partial\Omega} = \psi,\ |I_0| \leq M\}.$$

Besides, $\mathbb{D} = \mathcal{D}(A) \times H^2(\Omega) \times H^2(\Omega)$. We finally introduce the spaces $\mathbb{V}^\varepsilon_M \Subset \mathbb{D}^\varepsilon_M$ as

$$\mathbb{V}^\varepsilon_M = \mathbb{D}^\varepsilon_M \cap [H^3(\Omega) \times H^3(\Omega) \times H^3(\Gamma)],$$

with $\mathbb{V} = H^3(\Omega)\times H^3(\Omega)\times H^3(\Gamma)$. Concerning the nonlinearities $f, g \in C^3(\mathbb{R})$, we assume

$$\liminf_{|r|\to\infty} f'(r) > 0, \quad \liminf_{|r|\to\infty} g'(r) > 0 \tag{1.2}$$

$$f(v)v \geq \mu|v|^2 - \mu', \quad g(v)v \geq \mu|v|^2 - \mu'',\ \forall v \in \mathbb{R}, \tag{1.3}$$

for some $\mu > 0$ and $\mu', \mu'' \geq 0$. Here, (1.2) is a dissipativity condition which, in particular, implies the existence of a positive constant K such that $f', g' \geq -K$.

DEFINITION 1.1 *For any fixed $M > 0$, $T > 0$ and any triplet $z_0 = (w_0, u_0, \psi_0) \in \mathbb{D}^\varepsilon_M$, a solution to problem $\boldsymbol{P_\varepsilon}$ on the time interval $(0, T)$ is a triplet of functions $z(t) = (w(t), u(t), \psi(t)) \in C([0, T]; \mathbb{D}^\varepsilon_M)$ satisfying*

$$\begin{cases} \varepsilon w_t - \Delta w = -u_t, \qquad t > 0, \quad x \in \Omega \\ u_t - \Delta u + f(u) = w, \qquad t > 0, \quad x \in \Omega \\ \psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}} u + g(\psi) = 0, \qquad t > 0, \quad x \in \Gamma \\ \partial_{\boldsymbol{n}} w|_{\partial\Omega} = 0, \quad u|_{\partial\Omega} = \psi, \qquad t > 0, \quad x \in \Gamma \\ w|_{t=0} = w_0, \quad u|_{t=0} = u_0, \quad \psi|_{t=0} = \psi_0, \qquad x \in \Omega. \end{cases} \tag{1.4}$$

In what follows, we will denote by c a generic positive constant independent of ε which is allowed to vary even within the same formula; further dependencies will be made precise on occurrence. Besides, unless otherwise specified, every product is understood in the corresponding L^2-space. In particular, the norm and the scalar product in $L^2(\Omega)$ are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively, whereas the corresponding symbols in $L^2(\Gamma)$ are characterized by a subscript Γ.

2 A priori estimates

LEMMA 2.1 *There exist $\vartheta > 0$, a constant c and a monotone nonnegative function Q such that, for any fixed $\varepsilon \in (0,1]$, given any initial datum $z_0 \in \mathbb{D}^{\varepsilon}_M$, the following estimate holds for $z(t) = (w(t), u(t), \psi(t))$ solution to* (1.4)*:*

$$\|z(t)\|^2_{\mathbb{D}} + \|u_t(t)\|^2 + \|\psi_t(t)\|^2_\Gamma + \int_t^{t+1} [\varepsilon\|w_t(s)\|^2 + \|u_t(s)\|^2_{H^1} + \|\psi_t(s)\|^2_{H^1(\Gamma)}]ds \tag{2.1}$$
$$\leq Q\big(\|z_0\|^2_{\mathbb{D}}\big)e^{-\vartheta t} + c.$$

Proof. Throughout this proof, Q stands for a generic nonnegative increasing monotone function independent of ε and may vary even within the same formula. We first set $F(r) = \int_0^r f(s)ds$ and $G(r) = \int_0^r g(s)ds$. Then, we introduce the energy functional $E = E(t)$ defined by

$$E = \varepsilon\|w\|^2 + \|\nabla u\|^2 + \beta\|u\|^2 + \|\nabla_\Gamma \psi\|^2_\Gamma + (\lambda+\beta)\|\psi\|^2_\Gamma + 2\langle F(u), 1\rangle + 2\langle G(\psi), 1\rangle_\Gamma,$$

where $\beta > 0$ will be suitably chosen later. Multiplying the first equation of (1.4) by w and the second by $u_t + \beta u$, we have, on account of the boundary conditions,

$$\frac{dE}{dt} + 2\|\nabla w\|^2 + 2\beta\|\nabla u\|^2 + 2\beta\|\nabla_\Gamma\psi\|^2_\Gamma + 2\lambda\beta\|\psi\|^2_\Gamma + 2\|u_t\|^2 + 2\|\psi_t\|^2_\Gamma$$
$$= 2\beta\langle w, u\rangle - 2\beta\langle f(u), u\rangle - 2\beta\langle g(\psi), \psi\rangle_\Gamma.$$

Following [9], we write, in order to apply the Friedrich inequality,

$$2\langle w, u\rangle = 2\langle w - \langle w\rangle, u\rangle + 2|\Omega| I_\varepsilon\langle w\rangle - 2\varepsilon|\Omega|\langle w\rangle^2.$$

It is now readily seen that, for any $0 < \gamma < \beta$, the functional E satisfies the inequality

$$\frac{dE}{dt} + \gamma E = h,$$

where $h = h(t)$ is defined by

$$\begin{aligned} h = &-(2\beta-\gamma)(\|\nabla u\|^2 + \|\nabla_\Gamma\psi\|^2_\Gamma) - [\lambda(2\beta-\gamma) - \beta\gamma]\|\psi\|^2_\Gamma \\ &+ 2\gamma\big(\langle F(u) - f(u)u, 1\rangle + \langle G(\psi) - g(\psi)\psi, 1\rangle_\Gamma\big) \\ &+ 2(\gamma-\beta)\big(\langle f(u), u\rangle + \langle g(\psi), \psi\rangle_\Gamma\big) + \beta\gamma\|u\|^2 - \|\nabla w\|^2 \\ &- 2\|u_t\|^2 - 2\|\psi_t\|^2_\Gamma + 2\beta|\Omega| I_\varepsilon\langle w\rangle - 2(\beta-\gamma)\varepsilon|\Omega|\langle w\rangle^2 \\ &+ \gamma\varepsilon\|w\|^2 - 2\gamma\varepsilon|\Omega|\langle w\rangle^2 - \|\nabla w\|^2 + 2\beta\langle w - \langle w\rangle, u\rangle. \end{aligned}$$

By the Friedrich inequality, there exists a positive constant C_Ω such that

$$\|w - \langle w\rangle\|^2 = \|w\|^2 - |\Omega|\langle w\rangle^2 \le C_\Omega\|\nabla w\|^2.$$

This, together with the Young inequality, yields the following bound for the last line in the expression for h:

$$\begin{aligned}
&\gamma\varepsilon\|w\|^2 - 2\gamma\varepsilon|\Omega|\langle w\rangle^2 - \|\nabla w\|^2 + 2\beta\langle w - \langle w\rangle, u\rangle \\
&\le -(1-\gamma\varepsilon C_\Omega)\|\nabla w\|^2 + 2\beta\sqrt{C_\Omega}\|\nabla w\|\|u\| \\
&\le \frac{\beta^2 C_\Omega}{1-\gamma\varepsilon C_\Omega}\|u\|^2,
\end{aligned}$$

provided that $\gamma < 1/C_\Omega$. Besides, the following inequalities are proven in [18]:

$$\langle F(u) - f(u)u, 1\rangle \le K\|u\|^2 \quad \text{and} \quad \langle G(\psi) - g(\psi)\psi, 1\rangle_\Gamma \le K\|\psi\|_\Gamma^2,$$

for any $u \in L^2(\Omega)$ and $\psi \in L^2(\Gamma)$. Then, arguing as in [9], we have, if β and γ are small enough (and are independent of ε),

$$h \le \tilde{h} + r,$$

where

$$\begin{aligned}
r = &-\frac{1}{2}(2\beta-\gamma)(\|\nabla u\|^2 + \|\nabla_\Gamma\psi\|_\Gamma^2) - \frac{1}{2}[\lambda(\beta-\gamma) - \beta\gamma]\|\psi\|_\Gamma^2 \\
&+ 2\gamma\big(\langle F(u) - f(u)u, 1\rangle + \langle G(\psi) - g(\psi)\psi, 1\rangle_\Gamma\big) \\
&- (\beta-\gamma)\big(\langle f(u), u\rangle + \langle g(\psi), \psi\rangle_\Gamma\big) + \beta\gamma\|u\|^2 + \gamma\varepsilon\|w\|^2 \\
&- \|\nabla w\|^2 + 2\beta\langle w - \langle w\rangle, u\rangle - 2\gamma\varepsilon|\Omega|\langle w\rangle^2 \\
\le &-\frac{1}{2}(2\beta-\gamma)(\|\nabla u\|^2 + \|\nabla_\Gamma\psi\|_\Gamma^2) - \frac{1}{2}[\lambda(\beta-\gamma) - \beta\gamma]\|\psi\|_\Gamma^2 \\
&+ [2\gamma K - (\beta-\gamma)\mu](\|u\|^2 + \|\psi\|_\Gamma^2) + \beta\gamma\|u\|^2 + \frac{\beta^2 C_\Omega}{1-\gamma\varepsilon C_\Omega}\|u\|^2 + c \\
\le &\; c,
\end{aligned}$$

while

$$\begin{aligned}
\tilde{h} = &-\frac{1}{2}(2\beta-\gamma)c(\|u\|_{H^1}^2 + \|\psi\|_{H^1(\Gamma)}^2) - (\beta-\gamma)\big(\langle f(u), u\rangle + \langle g(\psi), \psi\rangle_\Gamma\big) \\
&- \|\nabla w\|^2 - 2\|u_t\|^2 - 2\|\psi_t\|_\Gamma^2 + 2\beta|\Omega|I_\varepsilon\langle w\rangle - 2(\beta-\gamma)\varepsilon|\Omega|\langle w\rangle^2.
\end{aligned}$$

We thus obtain

$$\frac{dE}{dt} + \gamma E \le \tilde{h} + c.$$

Next, integrating the first and the second equations of (1.4) over Ω, and the third over Γ, it follows that

$$\frac{d}{dt}(\varepsilon\langle w\rangle) + \langle w\rangle = \langle f(u)\rangle + \frac{|\Gamma|}{|\Omega|}\big(\langle\psi_t\rangle_\Gamma + \lambda\langle\psi\rangle_\Gamma + \langle g(\psi)\rangle_\Gamma\big). \qquad (2.2)$$

Hence, taking $\kappa = (1-\gamma\varepsilon)^{-1}(2\beta|\Omega|I_\varepsilon)$, it is apparent that the energy functional $E + \kappa\varepsilon\langle w\rangle$ satisfies the differential inequality

$$\begin{aligned}
&\frac{d}{dt}(E + \kappa\varepsilon\langle w\rangle) + \gamma(E + \kappa\varepsilon\langle w\rangle)\\
&\leq -\frac{1}{2}(2\beta-\gamma)c(\|u\|_{H^1}^2 + \|\psi\|_{H^1(\Gamma)}^2) - (\beta-\gamma)\big(\langle f(u), u\rangle + \langle g(\psi), \psi\rangle_\Gamma\big)\\
&\quad - \|\nabla w\|^2 - 2\|u_t\|^2 - 2\|\psi_t\|_\Gamma^2 - 2(\beta-\gamma)\varepsilon|\Omega|\langle w\rangle^2\\
&\quad + \kappa\langle f(u)\rangle + \kappa\frac{|\Gamma|}{|\Omega|}\big(\langle\psi_t\rangle_\Gamma + \lambda\langle\psi\rangle_\Gamma + \langle g(\psi)\rangle_\Gamma\big) + c.
\end{aligned}$$

Mimicking [9, (4.16)], there exist, for any arbitrarily small $\nu, \nu' > 0$, $C_\nu, C_{\nu'} > 0$ such that

$$|\langle f(u)\rangle| \leq \nu\langle f(u), u\rangle + C_\nu$$
$$|\langle g(\psi)\rangle_\Gamma| \leq \nu'\langle g(\psi), \psi\rangle_\Gamma + C_{\nu'},$$

which yields

$$\kappa\langle f(u)\rangle - (\beta-\gamma)\langle f(u), u\rangle \leq c,$$
$$\kappa\frac{|\Gamma|}{|\Omega|}\langle g(\psi)\rangle_\Gamma - (\beta-\gamma)\langle g(\psi), \psi\rangle_\Gamma \leq c.$$

Besides, it is readily seen that

$$\kappa\frac{|\Gamma|}{|\Omega|}\langle\psi_t\rangle_\Gamma - \|\psi_t\|_\Gamma^2 \leq c.$$

Thus, handling the other terms analogously, we end up with

$$\begin{aligned}
&\frac{d}{dt}(E + \kappa\varepsilon\langle w\rangle) + \gamma(E + \kappa\varepsilon\langle w\rangle)\\
&\qquad + \gamma'\big(\|u\|_{H^1}^2 + \|\psi\|_{H^1(\Gamma)}^2 + \|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|_\Gamma^2\big) \leq c,
\end{aligned}$$

and, since the energy functional satisfies the following inequalities:

$$E + \kappa\varepsilon\langle w\rangle \leq C_M\big(\varepsilon\|w\|^2 + \|u\|_{H^1}^2 + \|\psi\|_{H^1(\Gamma)}^2 + 2\langle F(u), 1\rangle + 2\langle G(\psi), 1\rangle_\Gamma + 1\big),$$
$$E + \kappa\varepsilon\langle w\rangle \geq C_M^{-1}\big(\varepsilon\|w\|^2 + \|u\|_{H^1}^2 + \|\psi\|_{H^1(\Gamma)}^2 + 2\langle F(u), 1\rangle + 2\langle G(\psi), 1\rangle_\Gamma - 1\big),$$

where the positive constant C_M is independent of ε, the Gronwall lemma gives

$$\begin{aligned}
&\varepsilon\|w(t)\|^2 + \|u(t)\|_{H^1}^2 + \|\psi(t)\|_{H^1(\Gamma)}^2 + 2\langle F(u(t)), 1\rangle + 2\langle G(\psi(t)), 1\rangle_\Gamma \qquad (2.3)\\
&\quad + \int_t^{t+1}[\|\nabla w(s)\|^2 + \|u_t(s)\|^2 + \|\psi_t(s)\|_\Gamma^2]ds\\
&\leq C_M e^{-\alpha t}[\varepsilon\|w_0\|^2 + \|u_0\|_{H^1}^2 + \|\psi_0\|_{H^1(\Gamma)}^2 + 2\langle F(u_0), 1\rangle + 2\langle G(\psi_0), 1\rangle_\Gamma] + C_M.
\end{aligned}$$

Next, we consider the problem formally obtained by differentiating the second and the third equations of (1.4) with respect to time, namely,

$$u_{tt} - \Delta u_t + f'(u)u_t = w_t, \qquad t > 0, \quad x \in \Omega \tag{2.4}$$

$$\psi_{tt} - \Delta_\Gamma \psi_t + \lambda\psi_t + \partial_{\boldsymbol{n}} u_t + g'(\psi)\psi_t = 0, \qquad t > 0, \quad x \in \Gamma, \tag{2.5}$$

supplemented with the initial conditions read from the problem

$$u_t(0) = \Delta u_0 - f(u_0) + w_0$$
$$\psi_t(0) = \Delta_\Gamma \psi_0 - \lambda\psi_0 - \partial_{\boldsymbol{n}} u_0 - g(\psi_0).$$

Multiplying (2.4) by u_t, (2.5) by ψ_t and the first equation of (1.4) by w_t, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|_\Gamma^2\big) + \varepsilon\|w_t\|^2 + \|\nabla u_t\|^2 + \|\nabla_\Gamma \psi_t\|_\Gamma^2 + \lambda\|\psi_t\|_\Gamma^2$$
$$+ \langle f'(u)u_t, u_t\rangle + \langle g'(\psi)\psi_t, \psi_t\rangle_\Gamma = 0.$$

Adding $(\|\nabla w\|^2 + \|u_t\|^2)/2$ to both sides of the above equality, we deduce from (1.2) that

$$\frac{d}{dt}\big(\|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|_\Gamma^2\big) + \|\nabla w\|^2 + \|u_t\|^2$$
$$+ 2[\varepsilon\|w_t\|^2 + \|\nabla u_t\|^2 + \|\nabla_\Gamma \psi_t\|_\Gamma^2 + \lambda\|\psi_t\|_\Gamma^2]$$
$$\leq c[\|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|_\Gamma^2]\,;$$

hence, in view of (2.3),

$$\|\nabla w(t)\|^2 + \|u_t(t)\|^2 + \|\psi_t(t)\|_\Gamma^2 + \int_t^{t+1} [\varepsilon\|w_t(s)\|^2 + \|\nabla u_t(s)\|^2 + \|\nabla_\Gamma \psi_t(s)\|_\Gamma^2]ds$$
$$\leq Q(\|z_0\|_{\mathbb{D}}^2)e^{-\alpha' t} + c. \tag{2.6}$$

To complete the H^1-norm of w, it is enough to observe that it follows from (2.2) that

$$\langle w(t)\rangle \leq Q(\|z_0\|_{\mathbb{D}}^2)e^{-ct} + c,$$

which, together with (2.6), yields

$$\|w(t)\|_{H^1}^2 \leq Q(\|z_0\|_{\mathbb{D}}^2)e^{-ct} + c.$$

We now consider the elliptic problem

$$\begin{cases} \Delta u - f(u) = h_1 \\ \Delta_\Gamma \psi - \lambda\psi - g(\psi) - \partial_{\boldsymbol{n}} u = h_2 \\ u|_\Gamma = \psi, \end{cases}$$

where $h_1 = u_t - w$ and $h_2 = \psi_t - \lambda\psi$. Since it follows from (2.3) and (2.6) that

$$\|h_1(t)\|^2 + \|h_2(t)\|_\Gamma^2 \le c + Q(\|z_0\|_{\mathbb{D}}^2)e^{-\gamma t}, \tag{2.7}$$

the maximum principle [10, Lemma A.2] applies, giving

$$\|u(t)\|_{L^\infty}^2 + \|\psi(t)\|_{L^\infty(\Gamma)}^2 \le c + Q(\|z_0\|_{\mathbb{D}}^2)e^{-\gamma t}.$$

This allows to view $f(u)$ and $g(\psi)$ as external forces as in [10, Theorem 1.1], entailing

$$\|u(t)\|_{H^2}^2 + \|\psi(t)\|_{H^2(\Gamma)}^2 \le c + Q\big(\|z_0\|_{\mathbb{D}}^2\big)e^{-\gamma t}.$$

So far, only the dissipativity in $\|\Delta w\|$ is missing, but, since we want to control this norm without ε, we rescale time as in [9, Lemma 1.3], that is, $t = \varepsilon\tau$. Then, the first equation of (1.4) reads

$$\tilde{w}_\tau - \Delta_N \tilde{w} = h(\tau) = -u_t(\varepsilon\tau),$$

and, owing to (2.6), we obtain at once

$$\|\tilde{w}(t)\|_{H^2}^2 \le Q(\|z_0\|_{\mathbb{D}}^2)e^{-ct} + c.$$

REMARK 2.1 Unfortunately, we are not able to derive these *a priori* estimates when (1.3) does not hold, since the phase-field model does not allow to argue as in [10], where a single equation is considered.

In the limiting case, we can prove the

LEMMA 2.2 *For any $z_0 \in \mathbb{D}_M^0$, if $z^0(t) = (w^0(t), u^0(t), \psi^0(t))$ solves* (1.4) *with $\varepsilon = 0$, then*

$$\|z^0(t)\|_{\mathbb{D}}^2 + \|u_t^0(t)\|^2 + \|\psi_t^0(t)\|_\Gamma^2 \le Q\big(\|z_0\|_{\mathbb{D}}^2\big)e^{-\vartheta t} + c. \tag{2.8}$$

Moreover, we have

$$\|w_t^0(t)\|^2 \le Q\big(\|z_0\|_{\mathbb{D}}^2\big)e^{-\vartheta t} + c. \tag{2.9}$$

Proof. Since (2.1) is uniform in ε, we can pass to the limit and we obtain at once (2.8). Next, (2.9) can be proved by arguing exactly as in [9, Lemma 1.8].

LEMMA 2.3 *For any pair of initial data $z_1, z_2 \in \mathbb{D}_M^\varepsilon$, there exist two positive constants C and L such that, if $z^i(t) = (w^i(t), u^i(t), \psi^i(t))$ is the solution originating from z_i, there holds*

$$\|z^1(t) - z^2(t)\|_{\mathbb{D}} + \|u_t^1(t) - u_t^2(t)\| + \|\psi_t^1(t) - \psi_t^2(t)\| \le Ce^{Lt}\|z_1 - z_2\|_{\mathbb{D}},$$

where the constants only depend on the norms of the initial data and are in particular independent of ε*.*

Proof. We can see that the difference $z(t) = z^1(t) - z^2(t) = (w(t), u(t), \psi(t))$ solves the problem

$$\begin{cases} \varepsilon w_t + u_t - \Delta w = 0 \\ u_t - \Delta u = w - \phi\, u \\ \psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}} u + \xi\psi = 0 \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0, \qquad u|_\Gamma = \psi \\ w(0) = w_0, \quad u(0) = u_0, \quad \psi(0) = \psi_0, \end{cases} \tag{2.10}$$

where $(w_0, u_0, \psi_0) = z_1 - z_2$ and

$$\phi = \phi(u^1, u^2) = \int_0^1 f'(su^1 + (1-s)u^2)ds,$$
$$\xi = \xi(\psi^1, \psi^2) = \int_0^1 g'(s\psi^1 + (1-s)\psi^2)ds.$$

Here, the integral mean is still conserved, but its modulus is now controlled as

$$|\langle u(t) + \varepsilon w(t)\rangle| = |\langle u_0 + \varepsilon w_0\rangle| \leq c,$$

with c possibly different from M, depending on the initial data. We will take advantage of Lemma 2.1 (respectively, of Lemma 2.2 when $\varepsilon = 0$), which, thanks to the assumptions on f and g, yields

$$\|\phi(t)\|_{H^2} + \|\phi_t(t)\| + \|\xi(t)\|_{H^2(\Gamma)} + \|\xi_t(t)\| \leq c.$$

Multiplying the first equation of (2.10) by w, the second by u_t and the third by ψ_t, we obtain

$$\frac{1}{2}\frac{d}{dt}[\varepsilon\|w\|^2 + \|\nabla u\|^2 + \|\nabla_\Gamma \psi\|_\Gamma^2 + \lambda\|\psi\|_\Gamma^2] + \|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|_\Gamma^2$$
$$= -\langle \phi u, u_t\rangle - \langle \xi\psi, \psi_t\rangle_\Gamma\,;$$

hence, by the Young inequality and an integration in time,

$$\varepsilon\|w(t)\|^2 + \|u(t)\|_{H^1}^2 + \|\psi(t)\|_{H^1(\Gamma)}^2 \leq e^{ct}\big(\varepsilon\|w_0\|^2 + \|u_0\|_{H^1}^2 + \|\psi_0\|_{H^1(\Gamma)}^2\big). \tag{2.11}$$

Differentiating with respect to time the second and the third equations in (2.10), we have

$$u_{tt} - \Delta u_t = w_t - \phi_t u - \phi\, u_t$$
$$\psi_{tt} - \Delta_\Gamma \psi_t + \lambda\psi_t + \partial_{\boldsymbol{n}} u_t + \xi_t\psi + \xi\psi_t = 0,$$

which, multiplied by u_t and ψ_t, respectively, and added to the product of the first equation of (2.10) by w_t, furnishes

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla w\|^2 + \|u_t\|^2 + \|\psi_t\|^2_\Gamma\big) + \varepsilon\|w_t\|^2 + \|\nabla u_t\|^2 + \|\nabla_\Gamma \psi_t\|^2_\Gamma + \lambda\|\psi_t\|^2_\Gamma$$
$$= -\langle \phi_t u, u_t\rangle - \langle \phi\, u_t, u_t\rangle - \langle \xi_t \psi, \psi_t\rangle_\Gamma - \langle \xi \psi_t, \psi_t\rangle_\Gamma.$$

Here, the assumptions on f and g immediately give

$$-\langle \phi\, u_t, u_t\rangle - \langle \xi\psi_t, \psi_t\rangle_\Gamma \le K[\|u_t\|^2 + \|\psi_t\|^2_\Gamma],$$

whereas

$$-\langle \phi_t u, u_t\rangle - \langle \xi_t\psi, \psi_t\rangle_\Gamma \le c[\|u\|_{H^1}\|u_t\|_{H^1} + \|\psi\|_{H^1(\Gamma)}\|\psi_t\|_{H^1(\Gamma)}].$$

Taking advantage of (2.11), these inequalities lead, in view of the initial conditions read from (2.10), to

$$\|\nabla w(t)\|^2 + \|u_t(t)\|^2 + \|\psi_t(t)\|^2_\Gamma \le ce^{ct}[\|w_0\|^2_{H^1} + \|u_0\|^2_{H^2} + \|\psi_0\|^2_{H^2(\Gamma)}].$$

Since

$$\langle w(t)\rangle = \langle u_t\rangle + \langle \phi u\rangle + \frac{|\Gamma|}{|\Omega|}[\langle \psi_t\rangle_\Gamma + \lambda\langle\psi\rangle_\Gamma + \langle \xi\psi\rangle_\Gamma],$$

it is straightforward to complete the H^1-norm of w. Then, the final estimates follow by standard parabolic regularity arguments.

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3 Existence of solutions

The main result of this section states as follows.

THEOREM 3.1 *For any $\varepsilon \in [0,1]$ and any $z_0 \in \mathbb{D}^\varepsilon_M$, there exists a unique solution $z(t)$ to problem $\boldsymbol{P_\varepsilon}$ which satisfies all the a priori estimates derived in the previous section.*

Since the limiting case is well known, we only focus on the case $\varepsilon \in (0,1]$. We set $\Omega_T = [0,T]\times\Omega$ and $\partial\Omega_T = [0,T]\times\partial\Omega$ and we will exploit the anisotropic Sobolev spaces $W^{1,2}_p(\Omega_T)$ and $W^{1,2}_p(\partial\Omega_T)$, constituted by functions that, together with their first time derivative and first and second space derivatives, belong to $L^p(\Omega_T)$ and $L^p(\partial\Omega_T)$, respectively (see, e.g., [8]). In what follows, we will need the embeddings $W^{1,2}_p(\Omega_T) \Subset C(\Omega_T)$ and $H^2(\Omega) \subset W_p^{2-\frac{2}{p}}(\Omega)$. The former compact inclusion follows from classical theorems, provided that $W^{2,p}(\Omega) \Subset C(\overline{\Omega})$, that is, when $2-3/p > 0$. The second embedding is satisfied if $2 \le p \le 10/3$. This lead us to confine $p \in (3, 10/3]$.

As a first step, we consider the linear nonhomogeneous problem for w and u with homogeneous boundary conditions, namely,

$$\begin{cases} \varepsilon w_t + u_t - \Delta w = h_1 \\ u_t - \Delta u - w = h_2 \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0, \qquad u|_\Gamma = 0 \\ w(0) = w_0, \qquad u(0) = u_0. \end{cases} \tag{3.1}$$

LEMMA 3.1 *For any fixed $\varepsilon > 0$, if $w_0, u_0 \in W_p^{2(1-\frac{1}{p})}(\Omega)$ and $h_1, h_2 \in L^p(\Omega_T)$, with $\langle h_1(t)\rangle = 0$, there exists a unique solution $(w,u) \in W_p^{1,2}(\Omega_T) \times W_p^{1,2}(\Omega_T)$ to (3.1) such that*

$$\begin{aligned} &\|w\|_{W_p^{1,2}(\Omega_T)} + \|u\|_{W_p^{1,2}(\Omega_T)} \\ &\le C\big(\|w_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)}\big), \end{aligned} \tag{3.2}$$

where C is a positive constant depending on T and ε, but is independent of w and u.

Proof. We proceed as in the proof of [10, Lemma 2.1]. Both equations in (3.1) are compact perturbations of the heat equation and, since the existence and uniqueness follow from standard arguments, we concentrate on the *a priori* estimates. For this purpose, it is convenient to determine u_t from the second equation, rewriting the first equation as

$$\varepsilon w_t + Aw = -\Delta u + h_1 - h_2. \tag{3.3}$$

Applying the L^p-theory to the second equation, we obtain

$$\|u\|_{W_p^{1,2}(\Omega_T)} \le c(\|h_2\|_{L^p(\Omega_T)} + \|w\|_{L^p(\Omega_T)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)}), \tag{3.4}$$

which, proceeding in a similar way for (3.3), implies the estimates

$$\begin{aligned} &\|w\|_{W_p^{1,2}(\Omega_T)} \\ &\le c(\|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)} + \|\Delta u\|_{L^p(\Omega_T)} + \|w_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)}) \\ &\le c(\|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)} + \|w\|_{L^p(\Omega_T)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|w_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)}). \end{aligned} \tag{3.5}$$

Using proper interpolation inequalities (see, e.g., [8, Chapter II, (3.2)] with $q = r = p$), we have

$$\|w\|_{L^p(\Omega_T)} \le c\|w\|_{L^\infty(0,T;L^2(\Omega))}^{1-\frac{2}{p}} \|w\|_{L^2(0,T;H^1(\Omega))}^{\frac{2}{p}}.$$

We thus accomplish our purpose provided that we can properly estimate $\|w\|_{L^\infty(0,T;L^2(\Omega))}$. Multiplying the first equation in (3.1) by w and the second by u_t and taking advantage of the null integral mean of h_1, we obtain

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|w\|^2 + \|\nabla u\|^2\big) + \|\nabla w\|^2 + \|u_t\|^2 = \langle h_1, w - \langle w\rangle\rangle + \langle h_2, u_t\rangle,$$

and the Friedrich and the Young inequalities entail the desired bound.

Next, we study the linear homogeneous problem with nonhomogeneous boundary conditions, but with null initial data, that is,

$$\begin{cases} \varepsilon w_t + u_t - \Delta w = 0 \\ u_t - \Delta u - w = 0 \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0, \qquad u|_\Gamma = \psi \\ w(0) = 0, \qquad u(0) = 0. \end{cases} \tag{3.6}$$

LEMMA 3.2 *For any fixed $\varepsilon > 0$, if $\psi \in W_p^{1-1/(2p),\,2-1/p}(\partial\Omega_T)$, there exists a unique solution $(w,u) \in W_p^{1,2}(\Omega_T) \times W_p^{1,2}(\Omega_T)$ to (3.6) such that*

$$\|w\|_{W_p^{1,2}(\Omega_T)} + \|u\|_{W_p^{1,2}(\Omega_T)} \le C\|\psi\|_{W_p^{1-1/(2p),\,2-1/p}(\partial\Omega_T)}, \tag{3.7}$$

for some positive constant C depending on T, but independent of ψ. Moreover,

$$\int_0^t \langle \partial_{\boldsymbol{n}} u(s), \psi(s)\rangle_\Gamma ds = \frac{1}{2}\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds \ge 0. \tag{3.8}$$

Proof. Following [10, Corollary 2.1], we consider the linear extension operator

$$T_p : W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T) \to W_p^{1,2}(\Omega_T) \quad \text{defined as} \quad (T_p\psi)|_{\partial\Omega_T} = \psi.$$

Besides, it is possible to construct this operator such that

$$\langle T_p\psi(t)\rangle = 0, \qquad \forall\psi \in W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T), \quad \forall t \ge 0.$$

Performing the change of variable $v = u - T_p\psi$, by straightforward computations we obtain that, if (w,u) solves (3.6), then (w,v) solves (3.1) with $h_1 = -\partial_t(T_p\psi)$, $h_2 = -\partial_t(T_p\psi) + \Delta(T_p\psi)$ and $v_0 = -T_p\psi_0$. By Lemma 3.1, we have the existence and uniqueness of the solution, together with estimate (3.7). In order to derive (3.8), it is enough to multiply the second equation in (3.6) by u and to integrate in time and space.

As a third step, we analyze the linearized version of problem $\boldsymbol{P_\varepsilon}$, that is,

$$\begin{cases} \varepsilon w_t + u_t - \Delta w = h_1 \\ u_t - \Delta u - w = h_2 \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0, \quad u|_\Gamma = \psi \\ \psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}} u + h_3 = 0 \\ w(0) = w_0, \qquad u(0) = u_0, \qquad \psi(0) = \psi_0. \end{cases} \tag{3.9}$$

LEMMA 3.3 *For any fixed $\varepsilon > 0$, if $h_1, h_2 \in L^p(\Omega_T)$, with $\langle h_1(t)\rangle = 0$, $h_3 \in L^p(\partial\Omega_T)$, $w_0, u_0 \in W_p^{2(1-\frac{1}{p})}(\Omega)$ and $\psi_0 \in W_p^{2(1-\frac{1}{p})}(\partial\Omega)$, then (3.9) possesses a unique solution $(w(t), u(t), \psi(t))$ such that*

$$\begin{aligned} &\|w\|_{W_p^{1,2}(\Omega_T)} + \|u\|_{W_p^{1,2}(\Omega_T)} + \|\psi\|_{W_p^{1,2}(\partial\Omega_T)} \\ &\le C(\|w_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|\psi_0\|_{W_p^{2(1-\frac{1}{p})}(\partial\Omega)} \\ &\quad + \|h_1\|_{L^p(\Omega_T)} + \|h_2\|_{L^p(\Omega_T)} + \|h_3\|_{L^p(\partial\Omega_T)}), \end{aligned} \tag{3.10}$$

for some constant $C > 0$ depending on T and ε, but independent of (w, u, ψ) and (h_1, h_2, h_3).

Proof. Since the existence, the uniqueness and the estimates for w are the same as above, we assume that w is fixed and we concentrate on the pair (u, ψ). The proof is similar to that of [10, Lemma 2.2], but we report it for the reader's convenience. There exists $\mathbb{T} : W_p^{1-\frac{1}{2p}, 2-\frac{1}{p}}(\partial\Omega_T) \to W_p^{1,2}(\Omega_T)$ solution operator to the following problem:

$$\begin{cases} v_t - \Delta v = w \\ v|_\Gamma = \psi \\ v(0) = 0. \end{cases} \tag{3.11}$$

Setting $v(t) = \mathbb{T}\psi(t)$ and $\theta(t) = u(t) - v(t)$, we obtain, in view of (3.9),

$$\begin{cases} \theta_t - \Delta\theta = h_2 \\ \theta|_{\partial\Omega} = 0 \\ \theta|_{t=0} = u_0 \\ \psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}} u + h_3 = 0. \end{cases} \tag{3.12}$$

In this new formulation of the problem, the unknowns are no longer coupled. Therefore, we consider the first three equations of (3.12), to which Lemma 3.1 applies, yielding

$$\|\theta\|_{W_p^{1,2}(\Omega_T)} \le C\big[\|u_0\|_{W_p^{2(1-\frac{1}{p})}(\Omega)} + \|h_2\|_{L^p(\Omega_T)}\big]. \tag{3.13}$$

Next, ψ solves

$$\begin{cases} \psi_t - \Delta_\Gamma \psi + \lambda\psi + \partial_{\boldsymbol{n}}(\mathbb{T}\psi) + \tilde{h} = 0 \\ \psi|_{t=0} = \psi_0, \end{cases} \tag{3.14}$$

where, due to (3.13), $\tilde{h} = h_3 + \partial_{\boldsymbol{n}}\theta \in L^p(\partial\Omega_T)$. Moreover, Lemma 3.2 and a suitable trace theorem provide

$$\|\partial_{\boldsymbol{n}}(\mathbb{T}\psi)\|_{L^p(\partial\Omega_T)} \le C\|\psi\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T)}.$$

Owing to this estimate, (3.14) is a compact perturbation of the heat equation on the boundary. Arguing as in [10], we then obtain the existence and uniqueness of solutions, as well as the estimate

$$\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} \le C\big[\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\psi\|_{W_p^{1-\frac{1}{2p},2-\frac{1}{p}}(\partial\Omega_T)} + \|\tilde{h}\|_{L^p(\partial\Omega_T)}\big].$$

The second term in the right-hand side of the above inequality can be handled by interpolation, yielding

$$\|\psi\|_{W_p^{1,2}(\partial\Omega_T)} \le C\big[\|\psi_0\|_{W_p^{2-\frac{2}{p}}(\partial\Omega)} + \|\psi\|_{L^2(\partial\Omega_T)} + \|\tilde{h}\|_{L^p(\partial\Omega_T)}\big].$$

Finally, we obtain, multiplying (3.14) by ψ and taking (3.8) into account,

$$\|\psi\|_{L^2(\partial\Omega_T)} \le C[\|\psi_0\|_{L^2(\partial\Omega)} + \|\tilde{h}\|_{L^2(\partial\Omega_T)}].$$

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. Given $z_0 = (w_0, u_0, \psi_0) \in \mathbb{D}_M^\varepsilon$, mimicking the proof of [10, Theorem 2.1], we consider the following homotopy of problem $\boldsymbol{P_\varepsilon}$:

$$\begin{cases} \varepsilon w_t - \Delta w + u_t = 0 \\ u_t - \Delta u - w = -sf(u) \\ \psi_t - \Delta_\Gamma\psi + \lambda\psi + \partial_{\boldsymbol{n}} u = -sg(\psi) \\ \partial_{\boldsymbol{n}} w|_\Gamma = 0, \quad u|_\Gamma = \psi \\ w(0) = w_0, \quad u(0) = u_0, \quad \psi(0) = \psi_0. \end{cases}$$

For any $s \in [0,1]$, this problem is equivalent to the following:

$$\begin{pmatrix} w \\ u \\ \psi \end{pmatrix} = \mathbb{M}_0 \begin{pmatrix} w_0 \\ u_0 \\ \psi_0 \end{pmatrix} + s\mathbb{M}_h \begin{pmatrix} 0 \\ -f(u) \\ -g(\psi) \end{pmatrix},$$

where $\mathbb{M}_0 : (w_0, u_0, \psi_0) \mapsto (w, u, \psi)$ is the solving operator of (3.9) with $h_1 = h_2 = h_3 = 0$ and $\mathbb{M}_h : (h_1, h_2, h_3) \mapsto (w, u, \psi)$ is the solving operator of (3.9) with null initial data. We now introduce the space $\Phi = W_p^{1,2}(\Omega_T) \times W_p^{1,2}(\Omega_T) \times W_p^{1,2}(\partial\Omega_T)$ which is compactly embedded into $C(\Omega_T) \times C(\Omega_T) \times C(\partial\Omega_T)$ and is such that the operator $(w, u, \psi) \mapsto \mathbb{M}_h\big(0, -f(u), -g(\psi)\big)$ is compact in Φ. Since each solution to the s-problem satisfies the *a priori* estimates uniformly in s, we obtain, by the Leray-Schauder theorem and arguing as in [10], the desired existence result. □

4 Dynamical Systems and Global Attractors

Collecting Lemmas 2.1 and 2.3 and Theorem 3.1, we can now state the

THEOREM 4.1 *Given any $\varepsilon \in [0,1]$ and any positive number M, problem $\boldsymbol{P_\varepsilon}$ generates a dissipative dynamical system $\{S_\varepsilon(t)\}$ on the phase-space $\mathbb{D}^\varepsilon_M$.*

Our next aim is to show the existence of an upper semicontinuous family of global attractors. We first have the

THEOREM 4.2 *Fixing $\varepsilon \in [0,1]$, the dynamical system $\{S_\varepsilon(t)\}$ on $\mathbb{D}^\varepsilon_M$ possesses the connected global attractor $\mathcal{A}_\varepsilon \subset \mathbb{V}^\varepsilon_M$. Besides, there exists a positive constant C such that, for any initial datum $z_0 \in \mathcal{A}_\varepsilon$, $S_\varepsilon(t)z_0 = (w(t), u(t), \psi(t))$ satisfies*

$$\varepsilon\|w_t(t)\|^2 + \|u_t(t)\|^2_{H^1} + \|\psi_t(t)\|^2_{H^1(\Gamma)} \le C, \quad \forall t \ge 0.$$

Proof. Given $z = (w_0, u_0, \psi_0) \in \mathbb{D}^\varepsilon_M$ we decompose the corresponding solution as $z(t) = z^d(t) + z^c(t)$, where

$$z^d(t) = (w^d(t), u^d(t), \psi^d(t)) \quad \text{and} \quad z^c(t) = (w^c(t), u^c(t), \psi^c(t))$$

solve

$$\begin{cases} \varepsilon w^d_t + u^d_t - \Delta w^d = 0 \\ u^d_t - \Delta u^d = w^d \\ \psi^d_t - \Delta_\Gamma \psi^d + \lambda\psi^d + \partial_{\boldsymbol{n}} u^d = 0 \\ \partial_{\boldsymbol{n}} w^d|_{\partial\Omega} = 0, \qquad u^d|_{\partial\Omega} = \psi^d \\ w^d(0) = w_0 - \langle w_0\rangle, \quad u^d(0) = u_0 - \langle u_0\rangle, \quad \psi^d(0) = \psi_0 \end{cases} \tag{4.1}$$

and

$$\begin{cases} \varepsilon w^c_t + u^c_t - \Delta w^c = 0 \\ u^c_t - \Delta u^c = w^c - f(u) \\ \psi^c_t - \Delta_\Gamma \psi^c + \lambda\psi^c + \partial_{\boldsymbol{n}} u^c + g(\psi) = 0 \\ \partial_{\boldsymbol{n}} w^c|_{\partial\Omega} = 0, \qquad u^c|_{\partial\Omega} = \psi^c \\ w^c(0) = \langle w_0\rangle, \quad u^c(0) = \langle u_0\rangle, \quad \psi^c(0) = 0, \end{cases} \tag{4.2}$$

respectively. Arguing exactly as in Lemma 2.1, it is not difficult to show that

$$\|z^d(t)\|_{\mathbb{D}} \le ce^{-\gamma t}\|z_0\|_{\mathbb{D}}.$$

Besides,

$$\|z^c(t)\|_{\mathbb{D}} \le \|z(t)\|_{\mathbb{D}} + \|z^d(t)\|_{\mathbb{D}} \le c, \qquad \forall t \ge 0,$$

and, in order to show the uniform boundedness of $z^c(t)$ in $\mathbb{V}$, it is enough to bound $\|\nabla\Delta w^c\|^2+\|\nabla\Delta u^c\|^2+\|\nabla_\Gamma\Delta_\Gamma\psi^c\|_\Gamma^2$ uniformly in time, which amounts to controlling the H^1-norms of the time derivatives of w^c, u^c and ψ^c. Thus, differentiating (4.2) with respect to time, we have

$$\begin{cases}\varepsilon w_{tt}^c + u_{tt}^c - \Delta w_t^c = 0\\ u_{tt}^c - \Delta u_t^c = w_t^c - f'(u)u_t\\ \psi_{tt}^c - \Delta_\Gamma \psi_t^c + \lambda\psi_t^c + \partial_{\boldsymbol{n}} u_t^c + g'(\psi)\psi_t = 0\\ \partial_{\boldsymbol{n}} w_t^c|_{\partial\Omega} = 0, \qquad u_t^c|_{\partial\Omega} = \psi_t^c.\end{cases} \tag{4.3}$$

Multiplying then the first equation of (4.3) by w_t^c, the second by u_{tt}^c and the third by ψ_{tt}^c, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|w_t^c\|^2+\|\nabla u_t^c\|^2+\|\nabla_\Gamma\psi_t^c\|_\Gamma^2+\lambda\|\psi_t^c\|_\Gamma^2\big)+\|\nabla w_t^c\|^2+\|u_{tt}^c\|^2+\|\psi_{tt}^c\|_\Gamma^2\\ &= -\langle f'(u)u_t, u_{tt}^c\rangle - \langle g'(\psi)\psi_t, \psi_{tt}^c\rangle_\Gamma\\ &\leq \|u_{tt}^c\|^2+\|\psi_{tt}^c\|_\Gamma^2+c\|u_t\|^2+c\|\psi_t\|_\Gamma^2.\end{aligned}$$

Finally, we deduce from Lemma 2.1 and the Gronwall lemma that

$$\|z^c(t)\|_{\mathbb{V}} \leq c, \qquad \forall t\geq 0$$

and the existence of the global attractor follows from classical results (see, e.g., [14]).

It is worth noting that the global attractor $\mathcal{A}_\varepsilon$ depends on the fixed constant M. Furthermore, we note that, in the limiting case $\varepsilon=0$, the first component reads $w^0(t)=\mathcal{J}(u^0(t))$ and $(S_0(t),\mathbb{D}_M^0)$ is a lifting of the dynamical system $(\mathbb{P}S_0(t), H^2(\Omega)\times H^2(\Gamma))$, where $\mathbb{P}$ is the projection $\mathbb{P}:\mathbb{D}_M^0\to H^2(\Omega)\times H^2(\Gamma)$ onto the second and the third components. In particular, $(\mathbb{P}S_0(t), H^2(\Omega)\times H^2(\Gamma))$ is dissipative and possesses the global attractor $\tilde{\mathcal{A}}_0\subset H^3(\Omega)\times H^3(\Gamma)$, hence

$$\mathcal{A}_0=\{(w,u,\psi)\in\mathbb{V}_M^0: \quad (u,\psi)\in\tilde{\mathcal{A}}_0, \quad w=\mathcal{J}(u)\}.$$

THEOREM 4.3 *The global attractor $\mathcal{A}_0$ is upper semicontinuous with respect to the family $\{\mathcal{A}_\varepsilon\}$, that is,*

$$\lim_{\varepsilon\to 0}\operatorname{dist}_{\mathbb{D}}(\mathcal{A}_\varepsilon,\mathcal{A}_0)=0,$$

where $\operatorname{dist}_{\mathbb{D}}$ *corresponds to the Hausdorff semidistance in* $\mathbb{D}$.

Proof. We adopt the procedure devised in [6]. Setting

$$\mathcal{A}=\bigcup_{\varepsilon\in(0,1]}\mathcal{A}_\varepsilon,$$

it is readily seen that $\mathcal{A}$ is a bounded subset of $H^3(\Omega) \times H^3(\Omega) \times H^3(\Gamma)$. We again consider the projection $\mathbb{P} : \mathbb{D}_M^\varepsilon \to H^2(\Omega) \times H^2(\Gamma)$ onto the second and the third components. First, notice that, owing to the definition of a continuous semigroup and to Theorem 4.2, the family

$$\mathcal{G} = \left\{ (u,\psi) \in C^0([0,\infty); H^2(\Omega) \times H^2(\Gamma)) \,\middle|\, \begin{array}{l} (u(t),\psi(t)) = \mathbb{P}S_\varepsilon(t) z_0 \\ z_0 \in \mathcal{A}_\varepsilon, \quad \varepsilon \in (0,1] \end{array} \right\}$$

is equicontinuous at zero. We will prove the upper semicontinuity by arguing by contradiction. Suppose that there exist $\rho > 0$ and two sequences $\varepsilon_n \to 0$ and $z_n \in \mathcal{A}_{\varepsilon_n}$ such that

$$\operatorname{dist}_{\mathbb{D}}(z_n, \mathcal{A}_0) \geq \rho. \tag{4.4}$$

We observe that each $z^n(t) = (w^n(t), u^n(t), \psi^n(t)) = S_{\varepsilon_n}(t) z_n \in \mathcal{A}_{\varepsilon_n}$ can be extended to a bounded complete trajectory of $S_{\varepsilon_n}(t)$. Besides, the set defined as

$$\bigcup_{t\in\mathbb{R}} \bigcup_{n\in\mathbb{N}} z^n(t) \subset \mathcal{A}$$

is a relatively compact subset of $\mathbb{D}$. On account of the equicontinuity of the family $\mathcal{G}$ and of the properties of the semigroups, the family $\mathbb{P}z^n : \mathbb{R} \to H^2(\Omega)\times H^2(\Gamma)$ is equicontinuous, allowing to apply the Ascoli theorem, which, together with a diagonalization procedure, leads to the existence of $(\hat{u},\hat{\psi}) \in C^0(\mathbb{R}; H^2(\Omega) \times H^2(\Gamma))$ such that $\mathbb{P}z^n \to (\hat{u},\hat{\psi})$ in $C^0([-N,N]; H^2(\Omega) \times H^2(\Gamma))$, for any $N > 0$, at least for a subsequence. Moreover,

$$\sup_{t\in\mathbb{R}} \|(\hat{u}(t),\hat{\psi}(t))\|_{H^2(\Omega)\times H^2(\Gamma)} < \infty.$$

We now show that $(\hat{u},\hat{\psi})$ is a bounded complete trajectory of $\mathbb{P}S_0(t)$. By definition,

$$\begin{cases} (\mathbb{I} - \Delta) w^n = -\varepsilon_n w_t^n + \Delta u^n - f(u^n) \\ u_t^n = w^n + \Delta u^n - f(u^n) \\ \psi_t^n - \Delta_\Gamma \psi^n + \lambda \psi^n + \partial_{\boldsymbol{n}} u^n + g(\psi^n) = 0 \\ \partial_{\boldsymbol{n}} w^n|_{\partial\Omega} = 0, \quad u^n|_{\partial\Omega} = \psi^n, \end{cases}$$

endowed with the obvious initial conditions. The convergence of $\mathbb{P}z^n$ and Theorem 4.2 yield that, up to a subsequence,

$$(w^n, u_t^n, \psi_t^n) \to (\mathcal{J}(\bar{u}), \mathcal{J}(\bar{u}) + \Delta\bar{u} - f(\bar{u}), \Delta_\Gamma\bar{\psi} - \lambda\bar{\psi} - \partial_{\boldsymbol{n}}\bar{u} - g(\bar{\psi}))$$

in $C^0([-N,N]; H^1(\Omega) \times (H^1(\Omega))^* \times (H^1(\Gamma))^*)$, for any $N > 0$. Besides, (w^n, u_t^n, ψ_t^n) converges to $(\bar{w}, \bar{u}_t, \bar{\psi}_t)$ in the sense of distributions and the pair $(\bar{u}(t), \bar{\psi}(t))$ is indeed a complete bounded trajectory of $\mathbb{P}S_0(t)$. This implies $(\bar{u}(0), \bar{\psi}(0)) \in \tilde{\mathcal{A}}_0$ and $(\bar{w}(0), \bar{u}(0), \bar{\psi}(0)) \in \mathcal{A}_0$, which, together with the convergence for $\mathbb{P}z^n(t)$, provides $\mathbb{P}z_n \to (\bar{u}(0), \bar{\psi}(0))$ in $H^2(\Omega) \times H^2(\Gamma)$. Finally, we know that $w^n \in C^0(\mathbb{R}, H^1(\Omega))$ and $\sup_{t\in\mathbb{R}} \|w^n(t)\|_{H^3} < \infty$, hence

$$w^n \to \bar{w} = \mathcal{J}(\bar{u}) \quad \text{in} \quad C^0([-N,N]; H^2(\Omega)),$$

for any $N > 0$. This allows to conclude that $z_n \to (\bar{w}(0), \bar{u}(0), \bar{\psi}(0)) \in \mathcal{A}_0$ in $\mathbb{D}$, against (4.4).

---

5 Exponential Attractors

We have the following exponential attractor's existence result [2].

THEOREM 5.1 *For any $\varepsilon \in [0,1]$, let $\mathcal{B}_\varepsilon$ be the absorbing ball given by Lemma 2.1 and Lemma 2.2 and let $t^* > 0$ be such that $S_\varepsilon(t)\mathcal{B}_\varepsilon \subset \mathcal{B}_\varepsilon$, for any $t \geq t^*$. Assume that the following conditions hold.*

(H1) Setting $S_\varepsilon(t^*) = S_\varepsilon$, the map S_ε satisfies, for every $z_1, z_2 \in \mathcal{B}_\varepsilon$,

$$S_\varepsilon z_1 - S_\varepsilon z_2 = L_\varepsilon(z_1, z_2) + K_\varepsilon(z_1, z_2),$$

where

$$\|L_\varepsilon(z_1, z_2)\|_{\mathbb{D}} \leq \kappa \|z_1 - z_2\|_{\mathbb{D}},$$

$$\|K_\varepsilon(z_1, z_2)\|_{\mathbb{V}} \leq \Lambda \|z_1 - z_2\|_{\mathbb{D}},$$

for some $\kappa \in (0, 1/2)$ and some $\Lambda > 0$ independent of ε.

(H2) The map

$$z \mapsto S_\varepsilon(t)z : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$$

is Lipschitz continuous on $\mathcal{B}_\varepsilon$, with a Lipschitz constant independent of $t \in [t^*, 2t^*]$ and of ε. Besides, the map

$$(t, z) \mapsto S_\varepsilon(t)z : [t^*, 2t^*] \times \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$$

is Hölder continuous, with an exponent independent of ε.

Then, there exists an exponential attractor $\mathcal{M}_\varepsilon$ on $\widetilde{\mathcal{B}}_\varepsilon = \overline{\mathcal{B}_\varepsilon}^{\mathbb{D}}$ that attracts $\widetilde{\mathcal{B}}_\varepsilon$ exponentially fast. Besides, the fractal dimension of $\mathcal{M}_\varepsilon$ is uniformly bounded with respect to ε.

Verification of (H1)

Given a pair of initial data $z_0^i \in \mathcal{B}_\varepsilon$, we set

$$z_0 = z_0^1 - z_0^2 = (w_0, u_0, \psi_0).$$

The difference of the corresponding solutions

$$z(t) = z^1(t) - z^2(t) = (w(t), u(t), \psi(t))$$

can be decomposed as $z(t) = \hat{z}^d(t) + \hat{z}^c(t)$, where

$$\hat{z}^d(t) = (\hat{w}^d(t), \hat{u}^d(t), \hat{\psi}^d(t)) \quad \text{and} \quad \hat{z}^c(t) = (\hat{w}^c(t), \hat{u}^c(t), \hat{\psi}^c(t))$$

solve (4.1) and

$$\begin{cases} \varepsilon \hat{w}^c_t + \hat{u}^c_t - \Delta \hat{w}^c = 0 \\ \hat{u}^c_t - \Delta \hat{u}^c = \hat{w}^c - \ell_1 u \\ \hat{\psi}^c_t - \Delta_\Gamma \hat{\psi}^c + \lambda \hat{\psi}^c + \partial_{\boldsymbol{n}} \hat{u}^c = -\ell_2 \psi \\ \partial_{\boldsymbol{n}} \hat{w}^c|_{\partial\Omega} = 0, \qquad \hat{u}^c|_{\partial\Omega} = \hat{\psi}^c \\ \hat{w}^c(0) = \langle w_0 \rangle, \quad \hat{u}^c(0) = \langle u_0 \rangle, \quad \hat{\psi}^c(0) = 0, \end{cases}$$

respectively. Here,

$$\ell_1(t) = \int_0^1 f'(su^1(t) + (1-s)u^2(t))ds$$
$$\ell_2(t) = \int_0^1 g'(s\psi^1(t) + (1-s)\psi^2(t))ds$$

satisfy, thanks to Lemma 2.1 (respectively, Lemma 2.2),

$$\|\ell_1(t)\|_{H^2} + \|\partial_t \ell_1(t)\| + \|\ell_2(t)\|_{H^2(\Gamma)} + \|\partial_t \ell_2(t)\|_\Gamma \le c, \qquad \forall t \ge 0.$$

Arguing as in Lemma 2.1, we see that

$$\|\hat{z}^d(t)\|_{\mathbb{D}} \le c e^{-\gamma t} \|z_0\|_{\mathbb{D}}.$$

We thus accomplish our purpose if we show that

$$\|\hat{z}^c(t)\|_{\mathbb{V}} \le c \|z_0\|_{\mathbb{D}},$$

for some c possibly depending on $t^\star$. This can be seen as in Theorem 4.2, by using Lemma 2.3 instead of Lemma 2.1. Finally, taking $t^\star$ large enough, the maps $L_\varepsilon(z_1, z_2) = \hat{z}^d(t^\star)$ and $K_\varepsilon(z_1, z_2) = \hat{z}^c(t^\star)$ satisfy (H1).

Verification of (H2)

Notice that, thanks to (2.10), only the Hölder continuity with respect to time is left to prove. Arguing as in [9, Lemma 3.3], we have, by interpolation,

$$\|v(t_2) - v(t_1)\|_{H^2} \le \|v(t_2) - v(t_1)\|_{H^3}^{2/3} \|v(t_2) - v(t_1)\|^{1/3}, \quad 2t^* \ge t_2 > t_1 \ge t^*.$$

Thus, $S_\varepsilon(\cdot)z$ is Hölder continuous with exponent 1/3, provided that

$$\sup_{z \in \mathcal{B}_\varepsilon} \|S_\varepsilon(t) z\|_{\mathbb{V}} \le c, \qquad \forall t \in [t^*, 2t^*].$$

By decomposing $S_\varepsilon(t)z$ as in the verification of (H1), it is possible to prove (cf. [9, Lemma 2.7]) that

$$\sup_{z\in\mathcal{B}_\varepsilon} \|S_\varepsilon(t)z\|_{\mathbb{V}} \le c(t), \qquad \forall t>0$$

and Lemma 2.1 implies, when $\varepsilon > 0$,

$$\begin{aligned}
&\|w(t_2)-w(t_1)\|^2 + \|u(t_2)-u(t_1)\|^2 + \|\psi(t_2)-\psi(t_1)\|_\Gamma^2 \\
&\le \frac{(t_2-t_1)}{\varepsilon}\int_0^{2t^\star} \{\varepsilon\|w_t(s)\|^2 + \|u_t(s)\|^2 + \|\psi_t(s)\|_\Gamma^2\}ds \\
&\le c(t_2-t_1),
\end{aligned}$$

where c now depends on ε. Analogously, when $\varepsilon = 0$, we deduce from Lemma 2.2

$$\begin{aligned}
&\|w(t_2)-w(t_1)\|^2 + \|u(t_2)-u(t_1)\|^2 + \|\psi(t_2)-\psi(t_1)\|_\Gamma^2 \\
&\le (t_2-t_1)\int_0^{2t^\star} \{\|w_t(s)\|^2 + \|u_t(s)\|^2 + \|\psi_t(s)\|_\Gamma^2\}ds \\
&\le c(t_2-t_1).
\end{aligned}$$

These two estimates yield the desired result.

REMARK 5.1 Since the global attractor is the minimal (for the inclusion) compact attracting set, we obtain $\mathcal{A}_\varepsilon \subset \mathcal{M}_\varepsilon$, which ensures the uniform (with respect to ε) boundedness of the fractal dimension of the global attractors.

REMARK 5.2 It would be interesting to construct a robust (i.e., upper and lower semicontinuous) family of exponential attractors for our problem. Indeed, it is in general very difficult to prove the lower semicontinuity of global attractors (this property may even be not valid). In contrast to this, this property is rather general for (proper) families of exponential attractors (see, e.g., [2], [3], [9] and [10]). Now, to do so, we need to study the boundary layer at $t = 0$ (cf. [15]; see also [9]). However, compared with the same problem with classical boundary conditions (see [9]), the dynamic boundary conditions yield new difficulties and we will come back to this problem in a forthcoming article.

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The model-problem associated to the Stefan problem with surface tension: an approach via Fourier-Laplace multipliers

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Abstract In this paper we introduce Fourier-Laplace multipliers and show how this technique can be used to investigate the model problem for the Stefan problem with surface tension.

1 Introduction

The classical Stefan problem has been studied by many authors for several decades. It is a model for phase transition in liquid-solid systems and accounts for heat diffusion and exchange of lateral heat in a homogeneous medium. For a precise formulation of the problem we refer to the monographs [20], [15] and Section 3.

It is known that the Stefan problem (without surface tension) admits a unique global weak solution provided that the given initial data have suitable signs. The existence of such a weak solution is closely related to the maximum principle. Regularity results for weak solutions for the one-phase Stefan problem were given for example in [4], [5], [8], [9], [13] and for the two-phase case for example in [2], [14], [17]. It is also known that many methods which were applied to the classical Stefan problem are not available for the Stefan problem with surface tension. In fact, the inclusion of surface tension will no longer allow to determine the phases merely by the sign of u. Existence of a global weak solution for the two-phase problem with surface tension was proved by Luckhaus in [12]; existence of a local classical solution was announced by the last author in [18] and [19] as a corollary of the estimates for the associated model problem. Recently it was proved by Escher, Prüss and Simonett [7]

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that the Stefan problem with surface tension admits a unique analytic solution under mild assumptions on the data in case that the interface is a graph over $\mathbb{R}^2$.

In this paper we study the linear model-problem associated to the two-phase Stefan problem with surface tension. Our basic tool to investigate this problem is Fourier-Laplace multipliers. By this approach we obtain a proof of the characterization of the existence of a strong solution of the linear model-problem by its data that is different from the one given in [7].

In Section 2 we show that these multipliers are closely related to classical Fourier multipliers and that results on L^p-boundedness of Fourier-Laplace multipliers can be obtained via results on boundedness of Fourier multipliers. In particular, we deduce a Fourier-Laplace multiplier theorem from the classical theorem due to Lizorkin [11].

Section 3 deals with the model-problem for the Stefan problem with surface tension. We show that the model-problem enjoys maximal L^p–regularity if the given data are in suitable Besov spaces.

2 The Fourier-Laplace transform

Fourier multipliers naturally occur in the study of elliptic differential equations in $\mathbb{R}^n$. This is one reason why Fourier multipliers have been studied for a long time (see [21], [11], [22]).

In order to deal in particular with parabolic equations the method of Fourier-Laplace multipliers turns out to be very helpful. We hence define in this section Fourier-Laplace multipliers and show how theorems on Fourier multipliers on $L^p(\mathbb{R}^{n+1})$ may be transferred to theorems on Fourier-Laplace multipliers on $L^p_\gamma(\Omega)$, where $\Omega = \mathbb{R}_+ \times \mathbb{R}^n$. Here

$$L^p_\gamma(\Omega) := \left\{ f \in L^1_{\mathrm{loc}}(\Omega) : \|f\|^p_{L^p_\gamma(\Omega)} := \int_{\mathbb{R}^n} \int_0^{+\infty} |\mathrm{e}^{-\gamma t} f(t,x)|^p \,\mathrm{d}t\,\mathrm{d}x < +\infty \right\}$$

for some constant $\gamma > 0$. The space $L^\infty_\gamma(\Omega)$ is defined in a similar way.

We start with the definition of the Fourier-Laplace transform. For $f \in C^\infty_c(\Omega)$ the *Fourier-Laplace transform* $\widehat{\widehat{f}}$ of f is defined by the following formula:

$$\widehat{\widehat{f}}(\lambda,\xi) = \int_0^{+\infty} \int_{\mathbb{R}^n} \mathrm{e}^{-ix\cdot\xi - \lambda t} f(t,x)\,dx\,dt, \quad \xi \in \mathbb{R}^n,\ \lambda \in \mathbb{C}.$$

Note that $\widehat{\widehat{f}}(\cdot,\xi)$ is an entire function for $\xi \in \mathbb{R}^n$ and integrating by parts, for $\gamma \in \mathbb{R}$, there exists $C_\gamma > 0$ such that

$$|\xi^{n+1}\widehat{\widehat{f}}(\lambda,\xi)| + |\lambda^2\widehat{\widehat{f}}(\lambda,\xi)| \le C_\gamma, \quad \xi \in \mathbb{R}^n,\ \mathrm{Re}\,\lambda \ge \gamma. \tag{2.1}$$

In order to introduce the notion of Fourier-Laplace multipliers, let $\mathbb{C}_{\gamma_0} := \{z \in \mathbb{C} : \operatorname{Re} z \geq \gamma_0\}$ for some $\gamma_0 \in \mathbb{R}$ and consider $m \in L^\infty(\mathbb{C}_{\gamma_0} \times \mathbb{R}^n)$. Assume that $m(\cdot, \xi)$ is analytic on $\mathbb{C}_{\gamma_0}$ for a.e. $\xi \in \mathbb{R}^n$. Then it follows from (2.1) that there exists $C > 0$ such that

$$|\xi^{n+1} m(\lambda, \xi) \widehat{\widetilde{f}}(\lambda, \xi)| + |\lambda^2 m(\lambda, \xi) \widehat{\widetilde{f}}(\lambda, \xi)| \leq C, \quad \text{a.e. } \xi \in \mathbb{R}^n, \ \operatorname{Re} \lambda \geq \gamma_0.$$

Therefore, by the inverse Fourier transform and by techniques similar to those described in [1, Theorem 2.5.1], for $f \in C_c^\infty(\Omega)$ and $\gamma \geq \gamma_0$ there exists $g \in L_\gamma^\infty(\Omega)$, given by

$$g(t, x) = \frac{1}{(2\pi)^{n+1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \mathrm{e}^{ix \cdot \xi + (\gamma + ir)t} m(\gamma + ir, \xi) \widehat{\widetilde{f}}(\gamma + ir, \xi) \, \mathrm{d}\xi \, \mathrm{d}r, \tag{2.2}$$

such that $m\widehat{\widetilde{f}} = \widehat{\widetilde{g}}$. Note that by Cauchy's theorem g is independent of γ.

Let $T_m : C_c^\infty(\Omega) \to L_{\gamma_0}^\infty(\Omega)$ denote the mapping $f \mapsto g$. We say that m is a *Fourier-Laplace multiplier* on $L_{\gamma_0}^p(\Omega)$ if T_m maps $C_c^\infty(\Omega)$ into $L_{\gamma_0}^p(\Omega)$ and there exists $C > 0$ such that

$$\|T_m f\|_{L_{\gamma_0}^p(\Omega)} \leq C \|f\|_{L_{\gamma_0}^p(\Omega)}, \quad f \in C_c^\infty(\Omega).$$

In this case, T_m can be extended to a bounded operator on $L_{\gamma_0}^p(\Omega)$.

By (2.2), we obtain that $T_m f$ admits for a.e. $x \in \mathbb{R}^n$ and $t > 0$ the representation

$$(T_m f)(t, x) = g(t, x) = \frac{1}{(2\pi)^{n+1}} \mathrm{e}^{\gamma_0 t} \mathcal{F}^{-1}[m(\gamma_0 + i\cdot, \cdot) \mathcal{F}(f_{\gamma_0})](t, x), \tag{2.3}$$

where $\mathcal{F}$ denotes the Fourier transform and

$$f_{\gamma_0}(t, x) := \begin{cases} \mathrm{e}^{-\gamma_0 t} f(t, x), & x \in \mathbb{R}^n, \quad t \geq 0, \\ 0, & x \in \mathbb{R}^n, \quad t < 0. \end{cases}$$

This representation allows us to transfer results on Fourier multipliers to Fourier-Laplace multipliers. For example, the following proposition may be deduced from Lizorkin's theorem on Fourier multipliers (see [11]).

PROPOSITION 2.1 *Let $1 < p < \infty$ and $m \in L^\infty(\mathbb{C}_{\gamma_0} \times \mathbb{R}^n)$ for some γ_0. Assume that $m(\cdot, \xi)$ is analytic on $\mathbb{C}_{\gamma_0}$ for a.e. $\xi \in \mathbb{R}^n$, $m(\gamma_0 + i\cdot, \cdot) \in C^{n+1}(\mathbb{R}^{n+1} \setminus \{0\})$ and there exists $M > 0$ such that*

$$|\xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \lambda^{\alpha_{n+1}} \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n} \partial_\lambda^{\alpha_{n+1}} m(\gamma_0 + i\lambda, \xi)| \leq M, \quad (\lambda, \xi) \in \mathbb{R}^{n+1} \setminus \{0\},$$

whenever $(\alpha_1, \ldots, \alpha_{n+1}) \in \{0, 1\}^{n+1}$. Then the function m is a Fourier-Laplace multiplier on $L_{\gamma_0}^p(\Omega)$.

Note that the analyticity of $m(\cdot,\xi)$ is needed for the representation (2.2) of g. Moreover, since the Fourier-Laplace transform $\widehat{\widehat{g}}(\cdot,\xi)$ of a function $g \in L^p_{\gamma_0}(\Omega)$ is analytic in $\{z \in \mathbb{C} : \text{Re } z > \gamma_0\}$ for a.e. $\xi \in \mathbb{R}^n$ it is not a strong restriction.

Furthermore, note that the second equality in (2.3) also holds for $t < 0$. Letting $\gamma \to \infty$ in (2.2), it thus follows that

$$\text{supp } \mathcal{F}^{-1}[m(\gamma_0 + i\cdot,\cdot)\mathcal{F}(f_{\gamma_0})] = \text{supp } g \subset \overline{\Omega}, \quad f \in C_c^\infty(\Omega). \tag{2.4}$$

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3 An Application to the Stefan problem

In this section we consider the linear model-problem associated to the classical Stefan problem with surface tension. The latter may be described as follows:

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Assume that Ω is filled with a liquid and solid phase, e.g., water and ice, which at time $t = 0$ cover the domains $\Omega_0^\pm$. The domains Ω_0^+ and Ω_0^- are separated by a compact surface Γ_0. For $t \geq 0$ let $\Gamma(t)$ denote the interface at time t and $\Omega^\pm(t)$ the domains of the two phases. We denote by $\nu(\cdot,t)$ the outer normal at $\Gamma(t)$ with respect to $\Omega^-(t)$. Furthermore, let $V(\cdot,t)$ be the normal velocity of the free boundary $\Gamma(t)$ and $H(\cdot,t)$ its mean curvature.

Let Γ_0 and $u_0^\pm : \Omega_0^\pm \to \mathbb{R}$ be given, where u_0^+ and u_0^- denote the initial temperatures of the liquid and solid phases. The strong formulation of the *two-phase Stefan problem* with surface tension consists of finding a family $\Gamma := \{\Gamma(t), t \geq 0\}$ of hypersurfaces and a family of functions $\{u^\pm : \cup_{t\geq 0}(\Omega^\pm(t) \times \{t\}) \to \mathbb{R}\}$ that satisfies

$$\begin{cases} \kappa^\pm \partial_t u^\pm = \mu^\pm \Delta u^\pm & \text{in } \Omega^\pm(t), \ \ t > 0, \\ u^\pm \qquad = \sigma H & \text{on } \Gamma(t), \quad t > 0, \\ [\mu\partial_\nu u] \quad = lV & \text{on } \Gamma(t), \quad t > 0, \\ u^\pm(0,\cdot) \ = u_0^\pm & \text{in } \Omega_0^\pm, \\ \Gamma(0,\cdot) \quad = \Gamma_0, & \end{cases} \tag{3.1}$$

where $l > 0$ is the latent heat, $\kappa^\pm > 0$ the heatcapacity, σ the surface tension and $\mu^\pm > 0$ the coefficient of heatconduction in the two phases. Moreover,

$$[\mu\partial_\nu] = \mu^+\partial_\nu u^+ - \mu^-\partial_\nu u^-$$

denotes the jump of the normal derivatives across the interface $\Gamma(t)$.

The investigation of the free boundary problems (3.1) is often reduced in a first step to the so-called *model-problem*. In our situation the model-problem reads as follows.

$$\begin{aligned} \partial_t u^\pm - \Delta u^\pm &= f, && (x,y) \in \mathbb{R}^3_\pm,\ t>0,\\ \partial_t \varrho - \partial_y u^+|_{y=0} + \partial_y u^-|_{y=0} &= g, && x \in \mathbb{R}^2,\ t>0,\\ u^\pm|_{y=0} + \Delta\varrho &= h, && x \in \mathbb{R}^2,\ t>0,\\ u^\pm|_{t=0} &= u_0^\pm, && (x,y)\in\mathbb{R}^3_\pm,\\ \varrho|_{t=0} &= 0, && x\in\mathbb{R}^2. \end{aligned} \tag{3.2}$$

Here, f, g, h and $u_0^\pm$ are the given data. For the time being, let $u_0^\pm = 0$ and $f = 0$. Then (3.2) reads as follows:

$$\begin{aligned} \partial_t u^\pm - \Delta u^\pm &= 0, && (x,y) \in \mathbb{R}^3_\pm,\ t>0,\\ \partial_t \varrho - \partial_y u^+|_{y=0} + \partial_y u^-|_{y=0} &= g, && x \in \mathbb{R}^2,\ t>0,\\ u^\pm|_{y=0} + \Delta\varrho &= h, && x \in \mathbb{R}^2,\ t>0,\\ u^\pm|_{t=0} &= 0, && (x,y)\in\mathbb{R}^3_\pm,\\ \varrho|_{t=0} &= 0, && x\in\mathbb{R}^2. \end{aligned} \tag{3.3}$$

It turns out that weighted Besov spaces are the right choice for the data g and h to ensure solvability of (3.3). For $1 < p < \infty$, $s \geq 0$, $\gamma \geq 0$ and $T > 0$ we define

$$B^{s/2,s}_{p,\gamma}((0,T)\times\mathbb{R}^2) := \Big\{u \in L^p((0,T)\times\mathbb{R}^2) : \|u\|_{B^{s/2,s}_{p,\gamma}((0,T)\times\mathbb{R}^2)} < +\infty\Big\},$$

where

$$\|u\|^p_{B^{s/2,s}_{p,\gamma}((0,T)\times\mathbb{R}^2)} := \int_0^T \|\mathrm{e}^{-\gamma t} f(t,\cdot)\|^p_{B^s_{p,p}(\mathbb{R}^2)}\,\mathrm{d}t + \int_{\mathbb{R}^2} \|\mathrm{e}^{-\gamma\cdot} f(\cdot,x)\|^p_{B^{s/2}_{pp}(0,T)}\,\mathrm{d}x.$$

Here, $B^s_{p,p}(\mathbb{R}^2)$ and $B^s_{p,p}(0,T)$ denote the usual Besov spaces (see [22] or [16]). The closure of $C^\infty_c((0,T)\times\mathbb{R}^2)$ in $B^{s/2,s}_{p,\gamma}((0,T)\times\mathbb{R}^2)$ is denoted by ${}_0B^{s/2,s}_{p,\gamma}((0,T)\times\mathbb{R}^2)$. For convenience, we set $B^{s/2,s}_p((0,T)\times\mathbb{R}^2) := B^{s/2,s}_{p,0}((0,T)\times\mathbb{R}^2)$.

Finally, for $T > 0$ we define

$$W^{1,2}_p((0,T)\times\mathbb{R}^3_\pm) := \Big\{u \in L^p((0,T)\times\mathbb{R}^3_\pm) : \|u\|_{W^{1,2}_p((0,T)\times\mathbb{R}^3_\pm)} < +\infty\Big\},$$

where

$$\|u\|^p_{W^{1,2}_p((0,T)\times\mathbb{R}^3_\pm)} := \int_0^T \|f(t,\cdot)\|^p_{W^{2,p}(\mathbb{R}^3_\pm)}\,\mathrm{d}t + \int_{\mathbb{R}^3_\pm} \|f(\cdot,x)\|^p_{W^{1,p}(0,T)}\,\mathrm{d}x.$$

Here, $W^{2,p}(\mathbb{R}^3_\pm)$ and $W^{1,p}(0,T)$ denote the usual Sobolev spaces.

In the next lemma we collect several properties of weighted Besov spaces.

LEMMA 3.1 *Let $\gamma > 0$, $T > 0$ and $1 < p < \infty$. Then the following assertions hold true.*

(a) *For $s > 0$ there exist an extension operator $E : B_p^{s/2,s}((0,T)\times\mathbb{R}^2) \to B_{p,\gamma}^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)$ and $c, C > 0$ such that for $u \in B_p^{s/2,s}((0,T)\times\mathbb{R}^2)$*

$$c\|Eu\|_{B_{p,\gamma}^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)} \le \|u\|_{B_p^{s/2,s}((0,T)\times\mathbb{R}^2)} \le C\|Eu\|_{B_{p,\gamma}^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)}.$$

(b) *Let $s \ge 0$ with $s-(1/p) \notin \mathbb{N}_0$, $r \le s$ and*

$$m_r : \begin{cases} \mathbb{R}^2\times\mathbb{C}_\gamma \to \mathbb{C} \\ (\xi,\lambda) \mapsto (\lambda+|\xi|^2)^{r/2} \end{cases}.$$

Then m_r is a Fourier-Laplace multiplier from ${}_0B_{p,\gamma}^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)$ into ${}_0B_{p,\gamma}^{(s-r)/2,\,s-r}(\mathbb{R}_+\times\mathbb{R}^2)$.

(c) *For $s \ge 0$ the space $B_p^{s/2,s}((0,T)\times\mathbb{R}^2)$ coincides with $B_{p,p}^{s/2}(0,T;L^p(\mathbb{R}^2))\cap L^p(0,T;B_{p,p}^s(\mathbb{R}^2))$.*

Proof. Observe first that for $s \in \mathbb{N}_0$ and $T > 0$, there exists $c, C > 0$ such that

$$c\|f\|_{W^{s,p}(0,T)} \le \|\mathrm{e}^{-\gamma\cdot}f\|_{W^{s,p}(0,T)} \le C\|f\|_{W^{s,p}(0,T)}, \quad f \in W^{s,p}(0,T).$$

Combined with real interpolation (see [22, equation 2.4.2(16), Theorem 4.3.1] this implies

$$c\|f\|_{B_{p,p}^s(0,T)} \le \|\mathrm{e}^{-\gamma\cdot}f\|_{B_{p,p}^s(0,T)} \le C\|f\|_{B_{p,p}^s(0,T)}, \quad f \in B_{p,p}^s(0,T). \qquad (3.4)$$

By [22, Theorem 4.2.2], there exists a strong $s/2$-extension operator $\tilde{E} : B_{pp}^l(0,T) \to B_{p,p}^l(\mathbb{R})$ for $0 \le l \le s/2$. We now define $E : B_{p,\gamma}^{s/2,s}((0,T)\times\mathbb{R}^2) \to B_p^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)$ by

$$(Ef)(x,t) = \big(\tilde{E}f(x,\cdot)\big)(t), \quad x \in \mathbb{R}^2,\ t > 0.$$

Then, by definition of E and (3.4), the operator E satisfies the inequality given in (a).

By Proposition 2.1, (2.4) and standard arguments in the theory of Besov spaces (see [3, Theorem 6.2.7 and Lemma 6.2.1] and [22, Theorem 2.10.3(b)]), we see that m_r is a Fourier-Laplace multiplier from ${}_0B_{p,\gamma}^{s/2,s}(\mathbb{R}_+\times\mathbb{R}^2)$ into ${}_0B_{p,\gamma}^{(s-r)/2,\,s-r}(\mathbb{R}_+\times\mathbb{R}^2)$ for $r \le 0$. In order to prove (b) for $0 < r \le \min\{2, s\}$ write

$$(\lambda+|\xi|^2)^{r/2} = \lambda^{r/2}\frac{(\lambda+|\xi|^2)^{r/2}}{\lambda^{r/2}+|\xi|^r} + |\xi|^r\frac{(\lambda+|\xi|^2)^{r/2}}{\lambda^{r/2}+|\xi|^r}$$

and observe that, by (2.4) and standard arguments as above,

$$(\lambda,\xi) \mapsto \frac{(\lambda+|\xi|^2)^{r/2}}{\lambda^{r/2}+|\xi|^r}$$

is a Fourier-Laplace multiplier from ${}_0B^{s/2,s}_{p,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)$ into ${}_0B^{s/2,s}_{p,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)$ and

$$(\lambda,\xi) \mapsto \lambda^{r/2} \quad \text{as well as} \quad (\lambda,\xi) \mapsto |\xi|^r$$

are Fourier-Laplace multipliers from ${}_0B^{s/2,s}_{p,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)$ into ${}_0B^{(s-r)/2,\, s-r}_{p,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)$. Finally, the case $\min\{2,s\} < r \le s$ follows by iteration.

We identify $v \in B^{s/2}_{p,p}(0,T;L^p(\mathbb{R}^2)) \cap L^p(0,T;B^s_{p,p}(\mathbb{R}^2))$ with $u \in B^{s/2,s}_p((0,T) \times \mathbb{R}^2)$ by $u(x,t) = (v(t))(x)$ for $(t,x) \in (0,T) \times \mathbb{R}^2$. Hence, (c) follows. □

We are now in the position to state the main result of this note. It was proved first by Escher, Prüss and Simonett in [7].

THEOREM 3.1 *Let $1 < p < \infty$ with $p \neq 3/2, 3$ and $T > 0$. Then there exists a unique solution $(u^\pm, \varrho)$ of problem (3.3) satisfying*

$$u^\pm \in W^{1,2}_p((0,T) \times \mathbb{R}^3_\pm), \quad \varrho \in B^{(3-(1/p))/2,\, 3-(1/p)}_p((0,T) \times \mathbb{R}^2),$$
$$\Delta\varrho \in B^{(2-(1/p))/2,\, 2-(1/p)}_p((0,T) \times \mathbb{R}^2)$$
$$\textit{and } (\partial_t\varrho)(0,\cdot) = 0 \textit{ in case } p > 3$$

if and only if the data f and g satisfy

$$g \in B^{(1-(1/p))/2,\, 1-(1/p)}_p((0,T) \times \mathbb{R}^2), \quad h \in B^{(2-(1/p))/2,\, 2-(1/p)}_p((0,T) \times \mathbb{R}^2)$$

and the compatibility conditions

$$h(0,\cdot) = 0, \qquad \textit{in case } p > \frac{3}{2},$$
$$g(0,\cdot) = 0, \qquad \textit{in case } p > 3.$$

Sketch of proof. The only if part follows from the trace theorem for Besov-spaces (see [16, 9.5.4], [22]) as in [7].

In order to prove the converse implication, we choose a strategy different from the one in [7] and make use of the Fourier-Laplace transform. In fact, taking Fourier-Laplace transforms in x and t of problem (3.3) we obtain

$$\begin{aligned} \lambda\widehat{\widetilde{u}}^\pm + |\xi|^2\widehat{\widetilde{u}}^\pm - \partial_y^2\widehat{\widetilde{u}}^\pm &= 0, \\ \lambda\widehat{\widetilde{\varrho}} - \partial_y\widehat{\widetilde{u}}^+|_{y=0} + \partial_y\widehat{\widetilde{u}}^-|_{y=0} &= \widehat{\widetilde{g}}, \\ \widehat{\widetilde{u}}^\pm|_{y=0} - |\xi|^2\widehat{\widetilde{\varrho}} &= \widehat{\widetilde{h}}. \end{aligned} \tag{3.5}$$

In order to solve this problem, let $v \in {}_0B^{(2-(1/p))/2,\, 2-(1/p)}_{p,\gamma}(\mathbb{R}_+ \times \mathbb{R}^2)$ and consider the problem

$$\begin{aligned} \lambda\widehat{\widetilde{u}}^\pm + |\xi|^2\widehat{\widetilde{u}}^\pm - \partial_y^2\widehat{\widetilde{u}}^\pm &= 0, \\ \widehat{\widetilde{u}}^\pm|_{y=0} &= \widehat{\widetilde{v}}. \end{aligned} \tag{3.6}$$

The solutions $\widehat{\widetilde{u}}^{\pm}$ of (3.6) are given by $\widehat{\widetilde{u}}^{\pm} = \mathrm{e}^{\mp y(\lambda+|\xi|^2)^{1/2}}\widehat{\widetilde{v}}$ and satisfy (see [10, Chapter IV, Theorem 9.1] or [6, Theorem 2.1])

$$\|u^{\pm}\|_{W_p^{1,2}((0,T)\times\mathbb{R}^3_{\pm})} \le C\|v\|_{B_p^{(2-(1/p))/2,\,2-(1/p)}((0,T)\times\mathbb{R}^2)}, \tag{3.7}$$

where C is independent of v. Moreover, we calculate that

$$\partial_y \widehat{\widetilde{u}}^{\pm}|_{y=0} = \mp(\lambda+|\xi|^2)^{1/2}\widehat{\widetilde{v}}.$$

It thus suffices to solve the problem

$$\begin{aligned} \lambda\widehat{\widetilde{\varrho}} + 2(\lambda+|\xi|^2)^{1/2}\widehat{\widetilde{v}} &= \widehat{\widetilde{g}}, \\ \widehat{\widetilde{v}} - |\xi|^2\widehat{\widetilde{\varrho}} &= \widehat{\widetilde{h}}, \end{aligned} \tag{3.8}$$

in order to find a solution to problem (3.3). The solution $(\widehat{\widetilde{v}}, \widehat{\widetilde{\varrho}})$ of (3.8) is given by

$$\begin{aligned} \widehat{\widetilde{v}} &= \frac{|\xi|^2}{S(\lambda,\xi)}\widehat{\widetilde{g}} + \frac{\lambda}{S(\lambda,\xi)}\widehat{\widetilde{h}}, \\ \widehat{\widetilde{\varrho}} &= \frac{1}{S(\lambda,\xi)}\widehat{\widetilde{g}} - 2\frac{(\lambda+|\xi|^2)^{1/2}}{S(\lambda,\xi)}\widehat{\widetilde{h}}, \end{aligned}$$

where

$$S(\lambda,\xi) = \lambda + 2(\lambda+|\xi|^2)^{1/2}|\xi|^2.$$

Let us define $\widehat{\widetilde{\mathcal{G}}} = (\lambda+|\xi|^2)^{(1-1/p)/2}\widehat{\widetilde{g}}$, $\widehat{\widetilde{\mathcal{H}}} = (\lambda+|\xi|^2)^{(2-1/p)/2}\widehat{\widetilde{h}}$, $\widehat{\widetilde{\mathcal{V}}} = (\lambda+|\xi|^2)^{(2-1/p)/2}\widehat{\widetilde{v}}$ and $\widehat{\widetilde{\mathcal{R}}} = (\lambda+|\xi|^2)^{(3-1/p)/2}\widehat{\widetilde{\varrho}}$. Then

$$\begin{aligned} \widehat{\widetilde{\mathcal{V}}} &= \frac{|\xi|^2(\lambda+|\xi|^2)^{1/2}}{S(\lambda,\xi)}\widehat{\widetilde{\mathcal{G}}} + \frac{\lambda}{S(\lambda,\xi)}\widehat{\widetilde{\mathcal{H}}}, \\ \widehat{\widetilde{\mathcal{R}}} &= \frac{(\lambda+|\xi|^2)}{S(\lambda,\xi)}\widehat{\widetilde{\mathcal{G}}} - 2\frac{(\lambda+|\xi|^2)}{S(\lambda,\xi)}\widehat{\widetilde{\mathcal{H}}}. \end{aligned}$$

By Proposition 2.1 and similar arguments as in Lemma 3.1(b),

$$\|\mathcal{V}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)} + \|\mathcal{R}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)} \le C(\|\mathcal{G}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)} + \|\mathcal{H}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)}),$$

where C is independent of $\mathcal{G}$ and $\mathcal{H}$. By definition of $\mathcal{V}$, $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{H}$ and Lemma 3.1(b), we thus obtain

$$\begin{aligned} &\|v\|_{{}_0B_{p,\gamma}^{(2-(1/p))/2,\,2-(1/p)}(\mathbb{R}_+\times\mathbb{R}^2)} + \|\rho\|_{{}_0B_{p,\gamma}^{(3-(1/p))/2,\,3-(1/p)}(\mathbb{R}_+\times\mathbb{R}^2)} \\ &\le C\Big(\|g\|_{{}_0B_{p,\gamma}^{(2-(1/p))/2,\,2-(1/p)}(\mathbb{R}_+\times\mathbb{R}^2)} + \|h\|_{{}_0B_{p,\gamma}^{(2-(1/p))/2,\,2-(1/p)}(\mathbb{R}_+\times\mathbb{R}^2)}\Big). \end{aligned}$$

In particular, we have $(\partial_t\varrho)(0,\cdot) = 0$ if $p > 3$. Lemma 3.1(a) now yields for any $T > 0$

$$v \in B_p^{(2-(1/p))/2,\,2-(1/p)}((0,T)\times\mathbb{R}^2), \quad \rho \in B_p^{(3-(1/p))/2,\,3-(1/p)}((0,T)\times\mathbb{R}^2),$$

and, therefore, by (3.7),

$$u \in W_p^{1,2}((0,T) \times \mathbb{R}^3_\pm).$$

So far we have shown the desired regularity for $u^\pm$ and ρ. It thus remains to prove that

$$\Delta\rho \in B_p^{(2-(1/p))/2,\, 2-(1/p)}((0,T) \times \mathbb{R}^2). \tag{3.9}$$

Let us introduce the function $\widehat{\widehat{\mathcal{P}}} = (\lambda + |\xi|^2)^{(2-(1/p))/2}(|\xi|^2\widehat{\widehat{\varrho}})$. Then

$$\widehat{\widehat{\mathcal{P}}} = \frac{|\xi|^2(\lambda + |\xi|^2)^{1/2}}{S(\lambda,\xi)}\widehat{\widehat{\mathcal{G}}} - 2\frac{|\xi|^2(\lambda + |\xi|^2)^{1/2}}{S(\lambda,\xi)}\widehat{\widehat{\mathcal{F}}},$$

and, as above, we obtain

$$\|\mathcal{P}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)} \le C(\|\mathcal{G}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)} + \|\mathcal{F}\|_{B^{0,0}_{p,\gamma}(\mathbb{R}_+\times\mathbb{R}^2)}).$$

Now, (3.9) follows from similar arguments as above. □

Let us now return to (3.2). For $f \in L^p((0,T)\times\mathbb{R}^3)$ and $u_0^\pm \in B^{2-(2/p)}_{p,p}(\mathbb{R}^3_\pm)$ the solution $u^\pm$ of the heat equation

$$\partial_t u^\pm - \Delta u^\pm = f, \qquad x \in \mathbb{R}^3_\pm,\ 0 < t < T,$$
$$u^\pm|_{t=0} = u_0^\pm, \quad x \in \mathbb{R}^3_\pm,$$

satisfies

$$\|u^\pm\|_{W^{1,p}((0,T)\times\mathbb{R}^3_\pm)} \le C\Big(\|f\|_{L^p((0,T)\times\mathbb{R}^3)} + \|u_0^\pm\|_{B^{2-(2/p)}_{p,p}(\mathbb{R}^3_\pm)}\Big),$$

where $C > 0$ is independent of f and $u_0^\pm$. We thus obtain the following corollary which also was proved first in [7].

COROLLARY 3.1 *Let $1 < p < \infty$ with $p \ne 3/2, 3$ and $T > 0$. Then there exists a unique solution $(u^\pm, \varrho)$ of problem (3.2) satisfying*

$$u^\pm \in W_p^{1,2}((0,T)\times\mathbb{R}^3_\pm), \quad \varrho \in B_p^{(3-(1/p))/2,\, 3-(1/p)}((0,T)\times\mathbb{R}^2) \text{ and}$$

$$\Delta\varrho \in B_p^{(2-(1/p))/2,\, 2-(1/p)}((0,T)\times\mathbb{R}^2)$$

$$\text{and } (\partial_t\varrho)(0,\cdot) = 0 \text{ in case } p > 3$$

if and only if the data f, g, h and $u_0^\pm$ satisfy

$$f \in L^p((0,T)\times\mathbb{R}^3), \quad u_0^\pm \in B^{2-(2/p)}_{p,p}(\mathbb{R}^3_\pm),$$

$$g \in B_p^{(2-(1/p))/2,\, 2-(1/p)}((0,T)\times\mathbb{R}^2), \quad h \in B_p^{(2-(1/p))/2,\, 2-(1/p)}((0,T)\times\mathbb{R}^2).$$

and the compatibility conditions

$$u_0^+|_{y=0} = u_0^-|_{y=0},$$

$$u_0^\pm|_{y=0} = h(0,\cdot), \qquad \textit{in case } p > \frac{3}{2},$$

$$-\partial_y u_0^+|_{y=0} + \partial_y u_0^-|_{y=0} = g(0,\cdot), \qquad \textit{in case } p > 3.$$

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The power potential and nonexistence of positive solutions

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Abstract We prove the nonexistence of positive solutions of the equation

$$D_t u = \mathrm{div}(|x|^{-2\gamma}\nabla u^m) + c|x|^{-2-2\gamma}u^m$$

for $0 < t < \varepsilon$ and x in a bounded domain in $\mathbb{R}^N$ containing origin. For suitable choices of $m < 1$ and all $\gamma > -1/2$, we show that positive solutions never exist provided $c > (N - 2\gamma - 2)^2/4$. That is, positive solutions never exist when the c exceeds the best constant for the Hardy inequality corresponding to the linear problem ($m = 1$).

1 Introduction

In quantum mechanics, the Schrödinger operator, $-\Delta + V(x)$, represents the energy. The kinetic energy operator, $-\Delta$, "scales like λ^2." By this we mean the following. Let $\lambda > 0$ and let U_λ be the unitary (on $L^2(\mathbb{R}^N)$) scaling operator defined by

$$U_\lambda f(x) = \lambda^{N/2} f(\lambda x).$$

That is, $U_\lambda f(x)$ is $f(\lambda x)$, normalized to have the same norm as f. Then $U_\lambda^{-1} = U_{1/\lambda}$ and "Δ scales like λ^2" means

$$U_\lambda^{-1}\Delta U_\lambda = \lambda^2\Delta. \tag{1.1}$$

Similarly, if $V(x) = |x|^{-\alpha}$ denotes the operator of multiplication by $|x|^{-\alpha}$, then

$$U_\lambda^{-1}|x|^{-\alpha}U_\lambda = \lambda^\alpha|x|^{-\alpha}, \tag{1.2}$$

so "$|x|^{-\alpha}$ scales like λ^2" iff $\alpha = 2$.

So consider the Hamiltonian with the inverse square potential,

$$\widetilde{H}_c = -\Delta - \frac{c}{|x|^2},$$

acting on $L^2(\mathbb{R}^N)$. When $\widetilde{H}_c$ is defined on the domain $D = C_c^\infty(\mathbb{R}^N)$ (or $D = C_c^\infty(\mathbb{R}^N \setminus \{0\})$ if $N = 1, 2$), then $\widetilde{H}_c$ is symmetric and $\widetilde{H}_c \geq 0$ iff $c \leq (N-2)^2/4$. This is the best constant in Hardy's inequality:

$$\left(\langle \widetilde{H}_c\varphi, \varphi\rangle = \right) \int_{\mathbb{R}^N} |\nabla\varphi|^2 dx - c\int_{\mathbb{R}^N} \frac{|\varphi|^2}{|x|^2} dx \geq 0$$

for all $\varphi \in D$ iff $c \leq (N-2)^2/4$. Now let H_c be the Friedrichs extension of $\widetilde{H}_c$ if $c \leq (N-2)^2/4$ and any selfadjoint extension otherwise. Since $\widetilde{H}_c$ is unitarily equivalent to $\lambda^2 \widetilde{H}_c$ for every $\lambda > 0$ by (1.1), (1.2), it follows that the spectrum of H_c is either $[0, \infty)$ or $\mathbb{R}$, and this holds according as $c \leq (\frac{N-2}{2})^2$ or $c > (\frac{N-2}{2})^2$.

This circle of ideas was the key for answering an old question of H. Brezis and J.-L. Lions. They assumed that $V \in L^\infty_{loc}(\mathbb{R}^N \setminus \{0\})$ was a positive potential with a singularity at the origin, and asked if the singularity could be so strong as to prevent a positive solution to

$$D_t u = \Delta u + V(x)u, \quad x \in \mathbb{R}^N, \quad t \geq 0 \tag{1.3}$$

from existing. This was settled by P. Baras and J. Goldstein [3] in 1984. They considered

$$\begin{cases} D_t u_n = \Delta u_n + V_n(x)u_n \\ u_n(x, 0) = f(x), \end{cases} \tag{1.4}$$

where $f \geq 0$ and

$$V_n(x) = \begin{cases} c/|x|^2 & \text{if } |x| \geq 1/n \\ cn^2 & \text{if } |x| \leq 1/n. \end{cases}$$

Then $V_n(x) = n^2 V_1(nx)$ scales nicely.

If f is not too big (e.g., $f \in \bigcup_{1 \leq p \leq \infty}(L^p(\mathbb{R}^N)))$, then the unique positive solution V_n to (1.4) exists on $\mathbb{R}^N \times [0, \infty)$. Moreover, $u_n(x, t)$ increases in n for each $x \in \mathbb{R}^N$ and $t > 0$. For $c \leq (N-2)^2/4$, $u_n(x, t)$ increases to $u(x, t)$, where u is the unique positive solution for $D_t u = \Delta u + c|x|^{-2}u$, $u(x, 0) = f(x)$.

Let us assume $f \in L^2(\mathbb{R}^N)$ for convenience. Then $u(x, t) = e^{-H_c} f(x)$ since $H_c \geq 0$. But u will exist in many other cases as well, even when f is a measure [3]. But for $c > (N-2)^2/4$, then

$$\lim_{n \longrightarrow \infty} u_n(x, t) = \infty$$

for all $x \in \mathbb{R}^N$ and all $t > 0$. This is "instantaneous blow up" [3].

There were various extensions of [3], for example, by replacing $\mathbb{R}^n$ by the Heisenberg group $\mathbb{H}^N$ [12], by replacing Δ by $\sum D_{x_i}(a_{ij}(x)D_{x_j})$, a uniformly elliptic operator with L^∞ coefficients [13], allowing potentials with large negative values [14], etc. But now we want to explain a significant contribution by X. Cabré and Y. Martel [4]. The idea is to determine exactly which condition on V prevents a positive solution of (1.3) from existing. The above result with the inverse square potential settled this when V is singular at only one point. Consider the Rayleigh quotient

$$\mathcal{R} = \frac{\int_\Omega |\nabla\phi|^2 dx - \int_\Omega V(x)\phi^2 dx}{\int_\Omega \phi^2 dx}$$

for real $\phi \in \widetilde{D} = C_c^\infty(\Omega \setminus K)$ where $\Omega \subset \mathbb{R}^N$ is open and K is a closed Lebesgue null subset of Ω. If $\Omega \neq \mathbb{R}^N$ then, roughly speaking, the following Cauchy-Dirichlet problem for $V(x) \geq 0$,

$$\begin{cases} D_t u = \Delta u + V(x)u, & x \in \Omega, t \geq 0, \\ u(x,0) = f(x) & x \in \Omega \\ u(x,t) = 0 & x \in \partial\Omega, t \geq 0 \end{cases}$$

has no positive solution if

$$Q_0 := \inf\{\mathcal{R}_\phi : \phi \in \widetilde{D} \setminus \{0\}\} = -\infty.$$

More precisely, there are lots of positive solutions if $Q_0 > -\infty$, and none exist if

$$\inf\left\{R_\phi + \frac{\varepsilon \int_\Omega |\nabla\phi|^2 dx}{\int_\Omega |\phi|^2 dx} : \phi \in \widetilde{D} \setminus \{0\}\right\} = -\infty$$

for some $\varepsilon > 0$. Thus the nonexistence set of V is "open" as was the case for the inverse square potential, when it was $((N-2)^2/4, +\infty)$ for V of the $V(x) = c|x|^{-2}$ with c varying. The work of [4] provided the background for extensions to nonlinear problems involving a general potential $V(x) \geq 0$. Several papers were then devoted to nonexistence of positive solutions for nonlinear parabolic equations, see for example [1], [2], [6], [7], [8], [9], [10], [11], [15], [16], [17].

We want to explain the ideas of our recent paper [8]. This we do in Section 2. This discussion leads to the detailed calculations in Section 3. These calculations form the heart of this paper. Section 4 concludes the paper with various remarks.

2 Nonexistence for nonlinear problems

Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Nonexistence of positive solutions was studied in [8] for the two problems

$$\begin{cases} D_t u = \operatorname{div}(|x|^{-2\gamma}\nabla u^m) + V(x)u^m & \text{in} \quad \Omega \times (0,T), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{2.1}$$

and

$$\begin{cases} D_t u = \operatorname{div}(|x|^{-\gamma p}|\nabla u|^{p-2}\nabla u) + V(x)u^{p-1} & \text{in} \quad \Omega \times (0,T), \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in} \quad \Omega. \end{cases} \tag{2.2}$$

When $m = 1, p = 2, \gamma = 0$ and $V(x) = c|x|^{-2}$, both of these problems reduce to the heat equation with inverse square potential. Let $\gamma = 0$ and $V \equiv 0$. Then (2.1) and (2.2) reduce respectively to the porous medium equation (or filtration equation, or fast diffusion equation) and the p-Laplace heat equation. When $\gamma \neq 0$ these are "weighted" equations with singularity at the origin.

Theorem 2.1 *Let $\gamma \in \mathbb{R}$, $N \geq 3$, $(N-2)/N \leq m < 1$, and $V(x) \in L^1_{loc}(\Omega \backslash \mathcal{K})$ where $\mathcal{K}$ is a closed Lebesgue null subset of Ω; if $\gamma \geq (N-2)/2$, we require that $0 \in \mathcal{K}$. Define*

$$\mathcal{R}^2_\phi = \frac{\int_\Omega (\varepsilon + |x|^{-2\gamma})|\nabla\phi|^2 dx - \int_\Omega V(x)|\phi|^2 dx}{\int_\Omega |\phi|^2 dx}.$$

If

$$\inf\{\mathcal{R}^2_\phi : 0 \neq \phi \in C^\infty_c(\Omega \setminus \mathcal{K})\} = -\infty,$$

for some $\varepsilon > 0$, then the problem (2.1) has no positive solution. When $N = 2$ (resp. $N = 1$), the condition on m should be replaced by $1/2 \leq m < 1$ (resp. $0 < m < 1$).

Theorem 2.2 *Let $\gamma \in \mathbb{R}$, $N \geq 2$, $\gamma \in \mathbb{R}$, $(2N)/(N+1) \leq p < 2$, $V(x) \in L^1_{loc}(\Omega \setminus \mathcal{K})$ where $\mathcal{K}$ is a closed Lebesgue null subset of Ω, and define*

$$\mathcal{R}^p_\phi = \frac{\int_\Omega (\varepsilon + |x|^{-p\gamma})|\nabla\phi|^p dx - \int_\Omega V(x)|\phi|^p dx}{\int_\Omega |\phi|^p dx}.$$

If

$$\inf\{\mathcal{R}^p_\phi : 0 \neq \phi \in C^\infty_c(\Omega \setminus \mathcal{K})\} = -\infty,$$

for some $\varepsilon > 0$, then the problem (2.2) has no positive solution. If $N = 1$ then the same conclusion holds provided p is assumed to satisfy $1 < p < 2$.

We conjecture that these results are valid for a wide choice of m and p, but the proof [8] only works for restricted values. The concrete examples associated with these theorems are as follows.

Theorem 2.3 *In Theorem 2.1, assume that* $0 \in \Omega$ *and* $V(x) = c|x|^{-2-2\gamma}$ *and* $\gamma > -1/2$. *Then (2.1) has no positive solutions provided* $c > C^*(N,\gamma) = (N-2\gamma-2)^2/4$.

Theorem 2.4 *In Theorem 2.2, assume that* $0 \in \Omega$ *and* $V(x) = c|x|^{-p-p\gamma}$ *and* $\gamma > -1/2$. *Then (2.2) has no positive solutions provided* $c > C^*(N,\gamma,p) = p^{-p}|N-p\gamma-p|^p$.

Note that $C^*(N,\gamma) = C^*(N,\gamma,2)$. In the above two theorems, $V(x)$ can be replaced by

$$\widetilde{V} = c|x|^{-p-\gamma p} + \beta|x|^{-p-\gamma p}\sin(|x|^{-\alpha})$$

for any $\alpha > 0$ and any real β provided $c > C^*(N,\gamma,p)$ with $p = 2$ for Theorem 2.3 and $\frac{2N}{N+1} < p \le 2$ for Theorem 2.4.

The key tool necessary for these results is the weighted version of Hardy's inequality due to Caffarelli, Kohn and Nirenberg [5]. Here is a complete version valid for all p, N and γ.

Theorem 2.5 *Let* $\alpha \in \mathbb{R}, 1 < p < \infty$. *Then*

$$\int_{\mathbb{R}^N} \frac{|\nabla u(x)|^p}{|x|^{\alpha-p}}dx \ge \left|\frac{N-\alpha}{p}\right|^p \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^\alpha}dx \tag{2.3}$$

for all $u \in W^{1,p}_{loc}(\mathbb{R}^N\setminus\{0\})$ *for which the right-hand side is finite. The constant*

$$C(N,\alpha,p) = \left|\frac{N-\alpha}{p}\right|^p$$

is the best possible in the sense that the inequality can fail to hold if $C(N,\alpha,p)$ *is replaced by any* $c > C(N,\alpha,p)$.

The proof is by scaling. See [8] for details.

In [8], we sketched very briefly how to show that Theorem 2.3 and 2.4 follows from Theorems 2.1 and 2.2. In the next section we give a detailed proof of Theorem 2.3. The proof of Theorem 2.4 is similar and is omitted.

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3 Detailed proof of Theorem 2.3

Let $\varepsilon > 0$ be given. Define the radial function $\phi_0 \in C_c(\Omega) \cap W^{1,\infty}(\Omega)$ by $\phi_0(x) = \phi(r)$ where $r = |x|$ and the radial function ϕ is given by

$$\phi(r) = \begin{cases} \varepsilon^{-a} & \text{if} \quad 0 \le r \le \varepsilon \\ r^{-a} & \text{if} \quad \varepsilon \le r \le 1 \\ 2-r & \text{if} \quad 1 \le r \le 2 \\ 0 & \text{if} \quad r \ge 2, \end{cases} \tag{3.1}$$

where $0 < \varepsilon < 1$ and $a > 0$. Then

$$\phi'(r) = \begin{cases} 0 & \text{if} \quad 0 \le r < \varepsilon \quad \text{and} \quad r > 2 \\ -ar^{-a-1} & \text{if} \quad \varepsilon < r < 1 \\ -1 & \text{if} \quad 1 < r < 2. \end{cases} \tag{3.2}$$

We are assuming that $0 \in \Omega \Subset \mathbb{R}^N$ and $N \ge 3$. Without loss of generality, we suppose $\bar{B}(0,2) = \{x \in \mathbb{R}^N : |x| \le 2\} \subset \Omega$; if not, we simply redefine ϕ, replacing 2 by R, where $\bar{B}(0,R) \subset \Omega$. This only results in notational changes in the proof that follows. Thus we have $\phi \in C_c(\Omega) \cap W^{1,\infty}(\Omega)$.

We want to show that, for some $\varepsilon_0 > 0$,

$$\inf_{0 \ne \phi \in C_c^\infty(\Omega \setminus \mathcal{K})} \frac{\int_\Omega (\varepsilon_0 + |x|^{-2\gamma})|\nabla\phi|^2 dx - \int_\Omega c|\phi|^2|x|^{-2-2\gamma} dx}{\int_\Omega |\phi|^2 dx} = -\infty \tag{3.3}$$

whenever $c > C^*(N,\gamma) = (N-2-2\gamma)^2/4$. We compute each integral in (3.3). First,

$$\int_\Omega |\phi|^2 dx = \omega_N \int_0^2 |\phi(r)|^2 r^{N-1} dr$$

where ω_N is the (Lebesgue) surface measure of the (N-1 dimensional) unit sphere in $\mathbb{R}^N$. Continuing, with ϕ as in (3.1),

$$\begin{aligned} \frac{1}{\omega_N}\int_\Omega |\phi|^2 dx &= \Big(\int_0^\varepsilon + \int_\varepsilon^1 + \int_1^2\Big)|\phi(r)|^2 r^{N-1} dr \\ &= \varepsilon^{-2a}\int_0^\varepsilon r^{N-1} dr + \int_\varepsilon^1 r^{N-2a-1} dr + \int_1^2 (2-r)^2 r^{N-1} dr. \end{aligned}$$

Therefore

$$\frac{1}{\omega_N}\int_\Omega |\phi|^2 dx = \frac{\varepsilon^{N-2a}}{N} - \frac{\varepsilon^{N-2a}}{N-2a} + K_1(a,N) \tag{3.4}$$

where

$$K_1(a,N) = \frac{1}{N-2a} + \frac{2^{N+2}-4}{N} - \frac{2^{N+3}-4}{N+1} + \frac{2^{N+4}-1}{N-2}.$$

If $0 < a < N/2$, then

$$\int_\Omega |\phi|^2 dx = K_1(a,N)(1+o(1)) \tag{3.5}$$

as $\varepsilon \longrightarrow 0^+$, and $K_1(a,N) > 0$. If $a > N/2$ then

$$\int_\Omega |\phi|^2 dx = \frac{2a}{(2a-N)N}\varepsilon^{N-2a}(1+o(1)) \tag{3.6}$$

as $\varepsilon \longrightarrow 0^+$. Thus far we assumed $a > 0$, $a \ne N/2$.

Next, we treat the case of $-1/2 < \gamma < (N-2)/2$. Then

$$\begin{aligned}\frac{1}{\omega_N}\int_\Omega \frac{\phi^2}{|x|^{2+2\gamma}}dx &= \int_0^2 \phi^2 r^{N-2\gamma-3}dr \\ &= \varepsilon^{N-2a-2\gamma-2}\Big(\frac{2a}{(N-2\gamma-2)(2a+2\gamma+2-N)}\Big) \\ &\quad + K_2(N,a,\gamma)\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}K_2(a,N,\gamma) =& \frac{1}{N-2a-2\gamma-2} + \frac{4(2^{N-2\gamma-2}-1)}{N-2\gamma-2} - \frac{4(2^{N-2\gamma-1}-1)}{N-2\gamma-1} \\ &+ \frac{2^{N-2\gamma}-1}{N-2\gamma};\end{aligned}$$

this is all valid when $-1/2 < \gamma < (N-2)/2$ and

$$a > \frac{N-2\gamma-2}{2}, \tag{3.8}$$

which we assume. Thus

$$\int_\Omega \frac{\phi^2}{|x|^{2+2\gamma}}dx = \omega_N \frac{2a\varepsilon^{N-2a-2\gamma-2}}{(N-2\gamma-2)(2a+2\gamma+2-N)}(1+o(1)) \tag{3.9}$$

as $\varepsilon \longrightarrow 0^+$.

Next,

$$\begin{aligned}\frac{1}{\omega_N}\int_\Omega \varepsilon_0|\nabla\phi|^2dx &= \varepsilon_0\Big\{\int_\varepsilon^1 + \int_1^2\Big\}(\phi'(r))^2 r^{N-1}dr \\ &= \ \varepsilon_0\big(\frac{a^2}{2a+2-N} + \frac{2^N-1}{2}\big)(1+o(1))\end{aligned} \tag{3.10}$$

as $\varepsilon \longrightarrow 0^+$ if $a < (N-2)/2$; otherwise,

$$\frac{1}{\omega_N}\int_\Omega \varepsilon_0|\nabla\phi|^2dx = \frac{\varepsilon_0 a^2\varepsilon^{N-2a-2}}{2a+2-N}(1+o(1)) \tag{3.11}$$

as $\varepsilon \longrightarrow 0^+$, if $a > (N-2)/2$. Recall $N \geq 3$. Our assumptions on a are $a > 0$, $a > (N-2-2\gamma)/2$, $a \neq N/2$, $a \neq (N-2)/2$.

Moreover,

$$\begin{aligned}\frac{1}{\omega_N}\int_\Omega r^{-2\gamma}|\nabla\phi|^2dx &= a^2\int_\varepsilon^1 r^{N-2a-2\gamma-3}dr + \int_1^2 r^{N-2\gamma-1}dr \\ &= a^2\Big(\frac{\varepsilon^{N-2a-2\gamma-2}}{2+2a+2\gamma-N}\Big)(1+o(1))\end{aligned} \tag{3.12}$$

as $\varepsilon \longrightarrow 0^+$ by (3.8).

Let

$$\mathcal{R} = \frac{\int_\Omega (\varepsilon_0 + |x|^{-2\gamma})|\nabla\phi|^2 dx - \int_\Omega c|\phi|^2|x|^{-2-2\gamma} dx}{\int_\Omega |\phi|^2 dx}. \tag{3.13}$$

First consider the case of $\gamma > 0$. Then using (3.5), (3.6), (3.10), (3.11),

$$\mathcal{R} = \frac{\varepsilon^{N-2a-2\gamma-2}(c_1(\varepsilon) + c_2 + c_3)}{c_0(\varepsilon)}(1 + o(1))$$

where

$$c_0(\varepsilon) = K_1(a, N) \qquad \text{if} \quad 0 < a < N/2, \tag{3.14}$$

$$c_0(\varepsilon) = \frac{2a}{(2a-N)N}\varepsilon^{N-2a} \qquad \text{if} \quad a > N/2, \tag{3.15}$$

and $0 < c_1(\varepsilon) = \varepsilon_0 K_0 \varepsilon^{b_1}$ or $\varepsilon_0 K_1 \varepsilon^{b_2}$, according as $a < (N-2)/2$ or $a > (N-2)/2$; here $K_0 = a^2(2a+2N)^{-1} + 2^{-1}(2^N - 1)$, $K_1 = a^2/(2a+2-N)$, $b_1 = 2 + 2a + 2\gamma - N > 0$, $b_2 = 2\gamma$.

Next, by (3.12) and (3.9),

$$c_2 = \frac{a^2}{2 + 2a + 2\gamma - N}, \qquad c_3 = \frac{-2ac}{(N - 2\gamma - 2)(2a + 2\gamma + 2 - N)}$$

(recall $\gamma < (N-2)/2$). For $0 < \gamma < (N-2)/2$ we conclude that $c_1(\varepsilon)+c_2+c_3 < 0$ for small $\varepsilon > 0$ if $c > (N-2-2\gamma)^2/4$ and $c_2 < -c_3$, i.e,

$$a < \frac{2c}{N - 2\gamma - 2}.$$

So by choosing

$$\frac{N-2-2\gamma}{2} < a < \frac{2c}{N-2-2\gamma}$$

(which is possible since $c > (N-2-2\gamma)^2/4$), we deduce $\mathcal{R} = c_1(\varepsilon) + c_2 + c_3 < -\varepsilon_1 < 0$ for small $\varepsilon > 0$. We also require $a \neq N/2, a \neq (N-2)/2$. In $\mathcal{R}$ (see (3.13)), the numerator is $\leq -\varepsilon_1 \varepsilon^{N-2a-2\gamma-2}$, and the denominator $c_0(\varepsilon)$ is given by either (3.14) or (3.15). Since $N - 2a - 2\gamma - 2 < 0$ and $N - 2a - 2\gamma - 2 < N - 2a$ (since $\gamma > 0$), it follows that $\lim_{\varepsilon \longrightarrow 0} \mathcal{R} = -\infty$. This completes the proof for the case of $0 \leq \gamma < (N-2)/2$. (We assumed $\gamma > 0$, but the proof works for $\gamma = 0$.)

For $\gamma \geq (N-2)/2$, we modify ϕ by defining ψ to be

$$\psi(r) = \begin{cases} \varepsilon^{-a-b} r^b & \text{if} \quad 0 \leq r \leq \varepsilon \\ \phi(r) & \text{if} \quad r \geq \varepsilon; \end{cases} \tag{3.16}$$

the choice of b will be made later. Then ψ defines a function (still denoted by ψ) in $C_c(\Omega \setminus \{x : |x| = \varepsilon\}) \cap W^{1,\infty}(\Omega)$ provided $b > 0$. (In (3.1), ϕ was the choice of ψ corresponding to $b = 0$.) As before we want to show that

$$\lim_{\varepsilon \longrightarrow 0^+} \mathcal{R} = -\infty$$

(see (3.3) and (3.13)), for suitable choices of a and b, if $c > (N-2\gamma-2)^2/4$. We next compute $\mathcal{R}$, using ψ rather than ϕ.

First,

$$\frac{1}{\omega_N}\int_\Omega |\psi|^2 dx = \frac{1}{\omega_N}\int_\Omega \phi^2 dx + \int_0^\varepsilon (\varepsilon^{-2a-2b} r^{2b+N-1} - \varepsilon^{2a} r^{N-1}) dr$$
$$= \frac{\varepsilon^{N-2a}}{2a-N} + K_1(a,N) + \frac{\varepsilon^{N-2a}}{2b+N}.$$

Therefore

$$\frac{1}{\omega_N}\int_\Omega |\psi|^2 dx = \begin{cases} K_1(a,N)(1+o(1)) & \text{if} \quad a < N/2, \\ K_2 \varepsilon^{N-2a}(1+o(1)) & \text{if} \quad a > N/2, \end{cases} \tag{3.17}$$

as $\varepsilon \longrightarrow 0^+$ (cf. (3.5), (3.7)), where

$$K_2 = \frac{2a+2b}{(2a-N)(2b+N)}.$$

Next,

$$\frac{1}{\omega_N}\int_\Omega \frac{\psi^2}{r^{2+2\gamma}} dx = \frac{1}{\omega_N}\int_\Omega \frac{\phi^2}{r^{2+2\gamma}} dx$$
$$+ \int_0^\varepsilon (\varepsilon^{2a-2b} r^{2b-2\gamma+N-3} - \varepsilon^{-2a} r^{-2\gamma+N-3}) dr$$
$$= \varepsilon^{N-2a-2\gamma-2}\left(\frac{1}{2a+2\gamma+2-N} + \frac{1}{2b+N-2\gamma-2}\right)$$
$$+ K_2(a,N,\gamma) \tag{3.18}$$

by (3.7) provided

$$b > \frac{2\gamma+2-N}{2} \ (\geq 0),$$

which we assume. Consequently $b > 0$ and

$$\frac{1}{\omega_N}\int_\Omega \frac{\phi^2}{|x|^{2+2\gamma}} dx = \frac{2(a+b)\varepsilon^{N-2a-2\gamma-2}}{(2a-N+2\gamma+2)(2b+N-2\gamma-2)}(1+o(1)) \tag{3.19}$$

as $\varepsilon \longrightarrow 0^+$, since $a > 0 \geq (N-2\gamma-2)/2$.

Next,

$$\frac{1}{\omega_N}\int_\Omega \varepsilon_0 |\nabla\psi|^2 dx = \frac{1}{\omega_N}\int_\Omega \varepsilon_0 |\nabla\phi|^2 dx + \int_0^\varepsilon \varepsilon^{-2a-2b} b^2 r^{2b+N-3} dr$$
$$= \frac{\varepsilon_0 a^2 \varepsilon^{N-2a-2}}{2+2a-N}(1+o(1)) + \frac{\varepsilon_0 b^2 \varepsilon^{N-2a-2}}{2b+N-2} \tag{3.20}$$
$$= \varepsilon_0 K_3 \varepsilon^{N-2a-2}(1+o(1))$$

as $\varepsilon \longrightarrow 0^+$, where

$$K_3 = \frac{a^2}{2+2a-N} + \frac{b^2}{2b+N-2} > 0;$$

recall $b > 0$ and assume $a > (N-2)/2 \geq 1/2$.

Next,

$$\begin{aligned}\frac{1}{\omega_N}\int_\Omega |x|^{-2\gamma}|\nabla\psi|^2 dx &= \frac{1}{\omega_N}\int_\Omega |x|^{-2\gamma}|\nabla\phi|^2 dx + \int_0^\varepsilon \varepsilon^{-2a-2b} b^2 r^{2b+N-3-2\gamma} dr \\ &= \frac{a^2\varepsilon^{N-2a-2\gamma-2}}{2+2a+2\gamma-N}(1+o(1)) + \frac{b^2\varepsilon^{N-2a-2\gamma-2}}{2b-2\gamma+N-2} \\ &\quad (\text{by}(3.9)) \\ &= \frac{c_4\varepsilon^{N-2a-2\gamma-2}}{(2a+2+2\gamma-N)(2b-2-2\gamma+N)}(1+o(1))\end{aligned} \tag{3.21}$$

as $\varepsilon \longrightarrow 0^+$, where

$$c_4 = a^2(2b-2\gamma+N-2) + b^2(2+2a+2\gamma-N). \tag{3.22}$$

Let

$$\mathcal{R}_\psi = \frac{\int_\Omega(\varepsilon_0 + |x|^{-2\gamma})|\nabla\psi|^2 dx - \int_\Omega c|\psi|^2|x|^{-2-2\gamma}dx}{\int_\Omega |\psi|^2 dx}. \tag{3.23}$$

By (3.17)–(3.22), we conclude that

$$\mathcal{R}_\psi = \frac{1}{c_0}\varepsilon^{N-2a-2\gamma-2}(c_5 + c_6(\varepsilon) + c_7)(1+o(1))$$

as $\varepsilon \longrightarrow 0^+$, where

$$c_0 = K_1(a,N) \quad \text{(as before)},$$

$$c_5 = \frac{-c(2a+2b)}{(2a+N-2\gamma-2)(2b+N-2\gamma-2)},$$

$$c_6(\varepsilon) = \varepsilon_0\left(\frac{a^2}{2+2a-N} + \frac{b^2}{2b+N-2}\right)\varepsilon^{2\gamma},$$

$$c_7 = \frac{c_4}{(2a+2+2\gamma-N)(2b-2-2\gamma+N)},$$

and c_4 is given by (3.22). To reach the desired conclusion that $\lim_{\varepsilon\longrightarrow 0^+}\mathcal{R}_\psi = -\infty$, it is enough to show that

$$c_5 + c_6(\varepsilon) + c_7 \leq -\delta_1 < 0$$

for some $\delta_1 > 0$. But $\gamma > 0$ implies $c_6(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$ for every $\varepsilon_0 > 0$. So it suffices to show $c_5 + c_7 < 0$. But this is equivalent to

$$c(2a+2b) > c_4,$$

that is

$$c > \frac{a^2(2b - 2\gamma + N - 2) + b^2(2a + 2\gamma + 2 - N)}{2a + 2b}. \tag{3.24}$$

Choose $b = (2\gamma - N + 2 + \delta_1)/2$ and $a = \delta/2$ for small $\delta > 0$. Then all of the construction on a and b are fulfilled, and (3.23) reduces to

$$\begin{aligned} c &> \frac{\frac{1}{4}\delta^2(-2N + 4 + \delta) + b^2(2\delta + 2\gamma + 2 - N)}{2\gamma - N + 2 + 2\delta} \\ &= b^2 + O(\delta^2) \end{aligned}$$

as $\delta \longrightarrow 0^+$. Then for small $\delta > 0$,

$$c > \Big(\frac{2\gamma - N + 2}{2}\Big)^2, \tag{3.25}$$

as desired. In other words, where c is chosen to satisfy (3.24), then (3.23) holds for small $\delta > 0$.

With (3.24) holding, it follows that

$$\lim_{\varepsilon \longrightarrow 0^+} \mathcal{R}_\psi = -\infty,$$

and the proof is complete.

4 Concluding Remarks

The condition that $\gamma > -1/2$ in Theorems 2.3 and 2.4 seems to be natural. We conjecture that it can be removed, but this perhaps requires suitable extensions of Theorems 2.1 and 2.2. We are studying this problem.

We have concentrated on the nonexistence aspect. For more related existence results see the paper [6] by Dall'Aglio, Giachetti and Peral.

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Inverse and direct problems for nonautonomous degenerate integrodifferential equations of parabolic type with Dirichlet boundary conditions[1]

Alfredo Lorenzi and Hiroki Tanabe

Abstract This paper deals with inverse and direct problems related to linear degenerate integrodifferential equations of parabolic type. The study of the direct problem is highly affected by the related inverse problems so that the results of the direct problems are just those needed to solve – locally in time – the inverse one. The latter is concerned with recovering – in a Hölder class – a memory kernel depending on time only.

1 Introduction and statement of the main result

This paper shows the deep links between *direct* and *inverse* problems related to the same equation – in this case a linear integrodifferential one.

The major part of this work will be devoted to a profound study – via Semigroup Theory – of the properties of the solution to the direct problem related to L^p-spaces, $p \in (1, 2)$, as far as the spatial variables are concerned. Yet, the route to be covered will be traced by a preliminary discussion of the inverse problem leading to an appropriate reformulation of the problem itself.

The reader is recommended to compare the results in the paper [1] in this book, dealing only with the direct problem for the same equation, but in a more general framework, to the ones more restrictive, but more specific, obtained here for the inverse problem, to understand which kind of additional requirements on the direct problem an inverse problem may give rise to.

Consequently, we devote this introduction to dealing with the problem of re-

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covering the scalar kernel $k : [0,T] \to \mathbf{R}$ in the following degenerate parabolic integrodifferential equation, Ω being a bounded open set of $\mathbf{R}^n$ with a C^2-boundary:

$$m(x,t)D_t u(x,t) + \mathcal{L}(t)u(x,t) + \int_0^t k(t-s)\mathcal{B}(s)u(x,s)\,ds$$
$$= f(x,t), \quad (x,t) \in \Omega \times [0,T], \qquad (1.1)$$

subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \qquad (1.2)$$

to the boundary conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0,T], \qquad (1.3)$$

and to the additional information

$$\Psi[m(\cdot,t)u(\cdot,t)] = g(t), \qquad t \in [0,T], \qquad (1.4)$$

Ψ being a linear continuous functional on $L^p(\Omega)$, $p \in (1,2)$.

Concerning function m and linear operators $\mathcal{L}(t)$, $\mathcal{B}(t)$, Ψ we make the following assumptions:

H1 $m = m_1 m_0$, where $m_1 \in C^{1+\rho}([0,T_0]; W^{2,\infty}(\Omega))$, $\rho \in (0,1)$, $m_0 \in C(\overline{\Omega})$ and $m_1(x,t) \geq c_0 > 0$, $m_1(x,0) = 1$ for all $(x,t) \in \overline{\Omega} \times [0,T_0]$, $m_0(x) > 0$ for a.e. $x \in \Omega$;

H2 $D_x m_0 \in C(\overline{\Omega})$ and $|D_x m_0(x)| \leq C m_0(x)^\delta$ for all $x \in \overline{\Omega}$ and some positive constant δ;

H3 $D_t^2 m \in C([0,T_0]; L^\infty(\Omega))$.

We emphasize that our problem is degenerate, since function m may vanish, but the set of its zeros is *time independent* due to assumption *H1* and coincides with the set of zeros of m_0.
Let now

$$\mathcal{L}(t) = -\sum_{i,j=1}^n D_{x_j}\left[a_{i,j}(x,t)D_{x_i}\right] + a(x,t), \qquad D_{x_j} = \frac{\partial}{\partial x_j},\ j = 1,\ldots,n,$$

be a linear second-order differential operator with real coefficients defined in $\overline{\Omega}$. We assume that the coefficients of $\mathcal{L}(t)$ satisfy

H4 $a_{i,j}, D_{x_i}a_{i,j}, a \in C^{0,1+\rho}(\overline{\Omega} \times [0,T_0])$, $\rho \in (0,1)$, $a_{i,j} = a_{j,i}$, $i,j = 1, \ldots, n$, $\sum_{i,j}^n a_{i,j}(x,t)\xi_i\xi_j \geq \mu|\xi|^2$, $a(x,t) \geq c_1 > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T_0]$ and $\xi \in \mathbf{R}^n$.

The realization of $\mathcal{L}(t)$ in $L^p(\Omega)$ with homogeneous Dirichlet boundary conditions is denoted by $L(t)$:

$$D(L(t)) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad L(t) = \mathcal{L}(t), \quad t \in [0, T_0].$$

Let

$$\mathcal{B}(t) = \sum_{i,j=1}^{n} b_{i,j}(x,t) D_{x_i} D_{x_j} + \sum_{i=1}^{n} b_i(x,t) D_{x_i} + b(x,t)$$

be a linear differential operator of order at most 2 for each $t \in [0, T_0]$. The coefficients of $B(t)$ are assumed to satisfy

H5 $b_{i,j}, b_i, b \in C^{0,1+\rho}(\overline{\Omega} \times [0, T_0]),\ i, j = 1, \ldots, n.$

The realization of $\mathcal{B}(t)$ in $L^p(\Omega)$ with homogeneous Dirichlet boundary conditions is denoted by $B(t)$:

$$D(B(t)) = W^{2,p}(\Omega), \quad B(t) = \mathcal{B}(t), \quad t \in [0, T_0].$$

To recover the unknown kernel k we need to introduce the new unknown

$$w(x,t) = D_t u(x,t) \quad \Longleftrightarrow \quad u(x,t) = u_0(x) + \int_0^t w(x,s)\,ds, \tag{1.5}$$

where $(x,t) \in \Omega \times (0, T]$.

By differentiation of equations (1.1) and (1.4) we easily deduce that the pair (w, k) solves the identification problem

$$D_t[m(x,t)w(x,t)] + L(t)w(x,t) = -L'(t)\Big[u_0(x) + \int_0^t w(x,s)\,ds\Big] - k(t)\mathcal{B}(0)u_0$$
$$- \int_0^t k(t-s)B(s)w(x,s)\,ds - \int_0^t k(t-s)B'(s)\Big(u_0(x) + \int_0^s w(\cdot,r)\,dr\Big)(x)\,ds$$
$$+ D_t f(x,t), \quad (x,t) \in \Omega \times [0, T], \tag{1.6}$$

$$m(x,0)w(x,0) = -\mathcal{L}(0)u_0(x) - f(x,0) := w_0(x), \quad x \in \Omega, \tag{1.7}$$

$$w(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T], \tag{1.8}$$

$$\Psi[m(\cdot,t)w(\cdot,t)] + \Psi\Big[D_t m(\cdot,t)\Big(u_0(x) + \int_0^t w(\cdot,s)\,ds\Big)\Big] = g'(t),\ t \in [0, T], \tag{1.9}$$

where we have set

$$D(L'(t)) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \qquad D(B'(t)) = W^{2,p}(\Omega), \quad t \in [0, T],$$

$$L'(t) = D_t\mathcal{L}(t) = -\sum_{i,j=1}^{n} D_{x_j}[D_t a_{i,j}(x,t) D_{x_i}] + D_t a(x,t),$$

$$B'(t) = -\sum_{i,j=1}^{n} D_t b_{i,j}(x,t) D_{x_i} D_{x_j} + \sum_{i=1}^{n} D_t b_i(x,t) D_{x_i} + D_t b(x,t).$$

Denote now by $M(t)$ the multiplication operator by $m(\cdot,t)$, $t \in [0,T_0]$. Then, according to assumption *H1*, $M(t)$ is formally invertible:

$$M(t)^{-1}u(x) = m(x,t)^{-1}u(x), \quad x \in \Omega,\ t \in [0,T_0],$$

but its inverse $M(t)^{-1}$ is *not* (in general) a bounded linear operator from $L^p(\Omega)$ into itself.

Introduce then the family of operators $\{A(t)\}_{t\in[0,T_0]}$ defined by

$$\begin{cases} D(A(t)) = \{m(\cdot,t)u : u \in D(L(t)) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\}, \\ A(t) = L(t)M(t)^{-1}. \end{cases}$$

We stress that, according to assumption *H1*, for all $t \in [0,T_0]$ we get

$$D(A(t)) = \{m_0 u : u \in D(L(t)) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\} =: D_0. \tag{1.10}$$

i.e., $D(A(t))$ is *independent* of $t \in [0,T_0]$. Moreover, D_0 is dense in $L^p(\Omega)$. Indeed, if $(m_0 u, z) = 0$ for all $u \in D_0$ and some $z \in L^{p'}(\Omega)$, then $m_0 z = 0$ a.e. in Ω. Hence $z = 0$ a.e. in Ω, since $m_0 > 0$ a.e. in Ω, by assumption *H1*. Under the previous assumption *H1* and the following one

H6 $a_{i,j}$, $D_{x_i}a_{i,j}$, $a \in C^{0,\rho}(\overline{\Omega} \times [0,T_0])$, $\rho \in (0,1]$, $a_{i,j} = a_{j,i}$, $i,j = 1,\dots,n$, $\sum_{i,j}^n a_{i,j}(x,t)\xi_i\xi_j \ge \mu|\xi|^2$, $a(x,t) \ge c_1 > 0$ for all $(x,t) \in \overline{\Omega} \times [0,T_0]$, and $\xi \in \mathbf{R}^n$,

we can show that operator $A(t) = L(t)M(t)^{-1}$ generates an infinitely differentiable semigroup $\{e^{-sA(t)}\}_{s>0}$ for any $t \in [0,T_0]$. Indeed, according to Theorem 2.1 in [2], holding under the more general assumption *H1*, the spectral equation

$$\lambda M(t)u(t) - L(t)u(t) = f$$

is solvable for any $\lambda \in \Sigma_1$, $t \in [0,T_0]$ and $f \in L^p(\Omega)$, where (cf. [2][p. 388] with $k_1(p) = \frac{1}{2}(p-1)^{1/2}|p-2|^{-1}$) $\Sigma_1 \subset \mathbf{C}$ is the sector

$$\Sigma_1 = \Big\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda + \frac{(p-1)^{1/2}}{4(2-p)}|\operatorname{Im}\lambda| + \frac{c_1}{8\|m\|_{L^\infty(\Omega)}}\Big\}, \quad p \in (1,2), \tag{1.11}$$

with half-width $\varphi \in (\pi/2, \pi - \frac{1}{2}(p-1)^{1/2}|p-2|^{-1})$. Moreover, $u(t)$ satisfies the generation estimate

$$\|M(t)u(t)\|_{L^p(\Omega)} \le \frac{C}{|\lambda|^\beta}\|f\|_{L^p(\Omega)}, \quad \lambda \in \Sigma_1,\ t \in [0,T_0], \tag{1.12}$$

with $\beta = 1/p$, the positive constant C depending on $(p, \|m\|_{C((\overline{\Omega}\times[0,T_0]))}, c_1)$, only.

If we require that function m_0 possesses the greater regularity listed in *H2*, we can increase the decay at infinity of the resolvent operator. More exactly, if $p \in (1,2)$, then, according to Theorem 4.1 in [2], we can choose, e.g.,

$$\beta = \max\Big\{\frac{1}{p}, \frac{1}{2-\delta}\Big\}. \tag{1.13}$$

Indeed, the quoted Theorem 4.1 holds under the more general assumptions *H1* and *H2*, since the following inequalities hold true for all $(x,t) \in \overline{\Omega} \times [0, T_0]$:

$$|D_x m(x,t)| \le |D_x m_1(x,t)| m_0(x) + m_1(x,t)|D_x m_0(x)|$$

$$\le [\|D_x m_1\|_{C(\overline{\Omega}\times[0,T_0])} \|m_0\|^{1-\delta}_{C(\overline{\Omega})} + C\|m_1\|_{C(\overline{\Omega}\times[0,T_0])}] m_0(x)^\delta.$$

Before stating our main result concerning our identification problem (cf. Theorem 1.1 stated at the end of the Section) we need the following assumption concerning the triplet (δ, p, ρ), introduced till now, as well as parameter γ related to the regularity exponent in Theorem 1.1:

H7 $\delta \in (0,1)$, $p \in (1,2)$, $\beta = \max\{1/p, 1/(2-\delta)\}$, $\rho \in (1-\beta, 1)$, $\rho/2 < \min\{2\beta - 1, \rho + \beta - 1\}$, $\gamma \in [\rho/2, \min\{2\beta - 1, \rho + \beta - 1\})$.

REMARK 1.1 Observe that $p \in (1,2)$ implies $\beta \in (1/2, 1)$. Assume first that $\rho + \beta - 1 \le 2\beta - 1$. This implies $\rho \in (1-\beta, \beta]$ and $\rho/2 < \rho + \beta - 1$, i.e., in turn, $2(1-\beta) < \rho \le \beta$ and $\beta \in (2/3, 1)$. Summing up, in this case we have

$$\beta \in (2/3, 1), \qquad 2 - 2\beta < \rho \le \beta, \qquad \gamma \in [\rho/2, \rho + \beta - 1).$$

On the contrary, if $2\beta - 1 < \rho + \beta - 1$, i.e., if $\beta < \rho < 1$, from $\rho/2 < 2\beta - 1$ we deduce $\beta < \rho < \min\{4\beta - 2, 1\}$. This inequality, in turn, implies $\beta \in (2/3, 1)$. Summing up, in this case we have

$$\beta \in (2/3, 1), \qquad \beta < \rho < \min\{4\beta - 2, 1\}, \qquad \gamma \in [\rho/2, 2\beta - 1).$$

We can now sharpen assumption *H3* to the following

H8 $D_t^2 m \in C^\gamma(\overline{\Omega} \times [0, T_0])$.

Finally, we assume that our data satisfy

H9 $f \in C^{1+\rho}([0,T_0]; L^p(\Omega))$, $g \in C^{2+\gamma}([0,T_0]; \mathbf{R})$, $u_0 \in D_0$, $w_0 := L(0)u_0 + f(\cdot, 0) \in D_0$, $B(0)u_0 \in (D_0, L^p(\Omega))_{\theta,p}$, $A(0)w_0 - D_t f(\cdot,0) + L'(0)u_0 \in (D_0, L^p(\Omega))_{\beta-\gamma,p}$ where the interpolation spaces $(D_0, X)_{\theta,p}$, $0 < \theta < 1$, $1 \le p \le \infty$, are defined by

$$(D_0, L^p(\Omega))_{\theta,p} = \Big\{ w(0) : t^{\theta - 1/p} w \in L^p((0,+\infty); D_0),$$
$$t^{\theta-1/p} w' \in L^p((0,+\infty); L^p(\Omega)) \Big\}. \tag{1.14}$$

We can now state our main result concerning our identification problem (1.1)–(1.4) related to $L^p(\Omega)$-spaces with $p \in (1,2)$.

THEOREM 1.1 *Under assumptions H1–H5, H7–H9 and the condition*

$$\Psi[B(0)u_0] \ne 0 \tag{1.15}$$

there exists a small enough $T \in (0, T_0]$ (cf. (6.36)) for which problem (1.1)–(1.4) admits a unique solution (u, k) with the properties: i) $m_0 u \in C^{1+\gamma}([0, T]; L^p(\Omega))$, ii) $u(t) \in D(L(t))$, $t \in [0, T]$, iii) $L(\cdot)u \in C^{\gamma}([0, T]; L^p(\Omega))$, iv) $k \in C^{\gamma}([0, T]; \mathbf{R})$.

We give now the plan of the paper. Section 2 is devoted to constructing the fundamental solution to equation (1.1) with $k = 0$, while Section 3 is concerned with the existence of a solution to the direct differential problem (1.1), with $k = 0$, (1.2), (1.3). The uniqueness of such a solution is proved in Section 4, while Section 5 is devoted to the regularization of the solution to the direct differential problem. Finally, the identification problem (1.1)–(1.4) is solved in Section 6.

2 Existence of the fundamental solution to the differential equation related to (1.1)

This section is first devoted to the construction of the fundamental solution $U(t, s)$ to the problem

$$\begin{aligned} &D_t v(t) + A(t)v(t) = f(t), \quad 0 < t \le T_0, \\ &v(0) = v_0 \end{aligned} \tag{2.1}$$

by a classical method. Then we will show the unique solvability of the initial value problem (2.1).

In this section we shall need the following weaker conditions *H10, H11* in the place of *H1, H4*:

H10 $m = m_1 m_0$, where $m_1 \in C^{\rho}([0, T_0]; W^{2,\infty}(\Omega))$, $\rho \in (0, 1]$, $m_0 \in C(\overline{\Omega})$ and $m_1(x, t) \ge c_0 > 0$, $m_1(x, 0) = 1$ for all $(x, t) \in \overline{\Omega} \times [0, T_0]$, $m_0(x) > 0$ for a.e. $x \in \Omega$;

H11 $a_{i,j}$, $D_{x_i} a_{i,j}$, $a \in C^{0,\rho}(\overline{\Omega} \times [0, T_0])$, $\rho \in (0, 1]$, $a_{i,j} = a_{j,i}$, $i, j = 1, \ldots, n$, $\sum_{i,j}^n a_{i,j}(x, t)\xi_i \xi_j \ge \mu |\xi|^2$, $a(x, t) \ge c_1 > 0$ for all $(x, t) \in \overline{\Omega} \times [0, T_0]$.

First we recall that, according to estimate (1.12), operators $A(t)$ satisfy the inequality

$$\|(\lambda + A(t))^{-1}\| \le C|\lambda|^{-\beta}, \quad \forall \lambda \in \Sigma_1, \ \forall t \in [0, T_0], \tag{2.2}$$

Σ_1 and β being defined by (1.11) and (1.13), respectively.

Hence

$$\|A(t)(\lambda + A(t))^{-1}\| \le C|\lambda|^{1-\beta}, \quad \forall \lambda \in \Sigma_1, \ \forall t \in [0, T_0], \tag{2.3}$$

for some positive constant C independent of $(\lambda, t) \in \Sigma_1 \times [0, T_0]$.

Making use of estimate (2.2), we can define, via the Dunford integral, the following family $\{e^{-rA(t)}\}_{t\in[0,T_0]}$ of linear semigroups:

$$e^{-rA(t)} = \frac{1}{2\pi i}\int_\Gamma e^{r\lambda}(\lambda + A(t))^{-1}\, d\lambda, \quad \forall r \in \mathbf{R}_+,\ \forall t \in (0, T_0], \tag{2.4}$$

where (throughout the paper)

$$\Gamma = \Gamma_- \cup \Gamma_+,\ \Gamma_- = \{re^{-i\eta} : r \geq 0\},\ \Gamma_+ = \{re^{i\eta} : r \geq 0\}, \quad \eta \in (\pi/2, \varphi).$$

The fundamental solution $U(t, s)$ to problem (1.3) will be searched for in the form

$$U(t, s) = e^{-(t-s)A(s)} + \int_s^t e^{-(t-\tau)A(\tau)}\Phi(\tau, s)d\tau, \quad 0 \leq s < t \leq T_0, \tag{2.5}$$

where

$$\Phi(t, s) + \Phi_1(t, s) + \int_s^t \Phi_1(t, \tau)\Phi(\tau, s)d\tau = 0, \tag{2.6}$$

$$\Phi_1(t, s) = (A(t) - A(s))e^{-(t-s)A(s)}. \tag{2.7}$$

By virtue of Theorem 2.1 in [2] one has

$$\|e^{-\tau A(t)}\| \leq C\tau^{\beta-1},\ \left\|D_\tau e^{-\tau A(t)}\right\| \leq C\tau^{\beta-2},\ \beta = \max\{1/p, 1/(2-\delta)\}, \tag{2.8}$$

where the norms of operators are to be meant in $\mathcal{L}(L^p(\Omega))$, $p \in (1, 2)$.

Consider now the following identities (cf. assumption *H10*), where $M_1(t)u = m_1(\cdot, t)u$ and $r, s, t \in [0, T_0]$:

$$(A(t) - A(s))A(r)^{-1} = (L(t)M_1(t)^{-1} - L(s)M_1(s)^{-1})M_1(r)L(r)^{-1}$$
$$= [L(t)L(r)^{-1}]L(r)[M_1(t)^{-1} - M_1(s)^{-1}]M_1(r)L(r)^{-1}$$
$$+[(L(t) - L(s))L(r)^{-1}]L(r)M_1(s)^{-1}M_1(r)L(r)^{-1}. \tag{2.9}$$

First we observe that, according to assumption *H10*, each operator $M_1(t)$, $t \in [0, T_0]$, is uniformly bounded and invertible with a bounded inverse operator $M_1(t)^{-1}$ such that the function $t \to M_1(t)^{-1}$ belongs to $C^\rho([0, T_0]; \mathcal{L}(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)))$.

Then we observe that, according to the existence and uniqueness result in [4], the family of linear elliptic operators $\{L(t)\}_{t\in[0,T_0]}$ endowed with Dirichlet boundary conditions consists of invertible operators.

Consequently, from (2.9), *H11* and well known results we obtain the estimate

$$\|(A(t) - A(s))A(r)^{-1}\| \leq C|t - s|^\rho, \quad r, s, t \in [0, T_0]. \tag{2.10}$$

As is easily seen, for $0 \le s < t \le T_0$, we get

$$\|\Phi_1(t,s)\| = \left\|(A(t)-A(s))A(s)^{-1}A(s)e^{-(t-s)A(s)}\right\| \le C(t-s)^{\rho+\beta-2}. \quad (2.11)$$

Hence, if $0 \le s < t \le T_0$,

$$\|\Phi(t,s)\| \le C(t-s)^{\rho+\beta-2}, \quad (2.12)$$

and

$$\left\|\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau\right\| \le C\int_s^t (t-\tau)^{\beta-1}(\tau-s)^{\rho+\beta-2}d\tau = C(t-s)^{\rho+2\beta-2}. \quad (2.13)$$

For $0 \le s < t \le T_0$ set

$$\begin{aligned} G(t,s) &= A(t)e^{-(t-s)A(t)} - A(s)e^{-(t-s)A(s)} \\ &= -\frac{1}{2\pi i}\int_\Gamma \lambda e^{\lambda(t-s)}\left((\lambda+A(t))^{-1} - (\lambda+A(s))^{-1}\right)d\lambda, \quad (2.14) \end{aligned}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} &\left\|(\lambda+A(t))^{-1} - (\lambda+A(s))^{-1}\right\| \\ &= \left\|(\lambda+A(t))^{-1}(A(t)-A(s))A(s)^{-1}A(s)(\lambda+A(s))^{-1}\right\| \\ &\le C|\lambda|^{-\beta}(t-s)^\rho|\lambda|^{1-\beta} = C(t-s)^\rho|\lambda|^{1-2\beta}, \quad 0 \le s < t \le T_0. \quad (2.15) \end{aligned}$$

Hence one derives

$$\begin{aligned} \|G(t,s)\| &\le C\int_\Gamma |\lambda|e^{\mathrm{Re}\lambda(t-s)}(t-s)^\rho|\lambda|^{1-2\beta}|d\lambda| \\ &= C(t-s)^\rho\int_\Gamma |\lambda|^{2-2\beta}e^{-\mathrm{Re}\lambda(t-s)}|d\lambda| \\ &= C(t-s)^\rho(t-s)^{2\beta-3} = C(t-s)^{\rho+2\beta-3}, \quad 0 \le s < t \le T_0. \quad (2.16) \end{aligned}$$

Also using (2.15) it is easy to show that

$$\|e^{-\tau A(t)} - e^{-\tau A(s)}\| \le C(t-s)^\rho\tau^{2\beta-2}, \quad \tau > 0,\ 0 \le s < t \le T_0. \quad (2.17)$$

Let now $0 \le s < \tau < t \le T_0$. Since

$$A(s)(e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)}) = -\int_\tau^t A(s)^2e^{-(r-s)A(s)}dr$$

$$= -\frac{1}{2\pi i}\int_\tau^t dr\int_\Gamma \lambda^2e^{\lambda(r-s)}(\lambda+A(s))^{-1}d\lambda,$$

$$\left\|\frac{1}{2\pi i}\int_\Gamma \lambda^2e^{\lambda(r-s)}(\lambda+A(s))^{-1}d\lambda\right\|$$

$$\le C\int_\Gamma |\lambda|^2e^{\mathrm{Re}\lambda(r-s)}|\lambda|^{-\beta}|d\lambda| \le C(r-s)^{\beta-3},$$

one has

$$
\begin{aligned}
&\Big\|A(s)(e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)})\Big\| \\
&\le C\int_\tau^t (r-s)^{\beta-3}\,dr \le C(\tau-s)^{\beta-1}\int_\tau^t (r-s)^{-2}dr \\
&= C(\tau-s)^{\beta-1}\frac{t-\tau}{(\tau-s)(t-s)} = C\frac{t-\tau}{t-s}(\tau-s)^{\beta-2}. \qquad (2.18)
\end{aligned}
$$

Therefore, from the following identities, where $0 \le s < \tau < t \le T_0$,

$$
\begin{aligned}
&\Phi_1(t,s) - \Phi_1(\tau,s) \\
&\quad = (A(t)-A(\tau))e^{-(t-s)A(s)} + (A(\tau)-A(s))(e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)}) \\
&\quad = (A(t)-A(\tau))A(s)^{-1}A(s)e^{-(t-s)A(s)} \\
&\qquad + (A(\tau)-A(s))A(s)^{-1}A(s)(e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)}),
\end{aligned}
$$

one deduces

$$
\|\Phi_1(t,s) - \Phi_1(\tau,s)\| \le C\Big\{(t-\tau)^\rho(t-s)^{\beta-2} + \frac{t-\tau}{t-s}(\tau-s)^{\rho+\beta-2}\Big\}. \quad (2.19)
$$

Moreover, if $v_0 \in D_0$, from the identity

$$
\begin{aligned}
[\Phi_1(t,s) - \Phi_1(\tau,s)]v_0 &= (A(t)-A(\tau))A(s)^{-1}e^{-(t-s)A(s)}A(s)v_0 \\
&+(A(\tau)-A(s))A(s)^{-1}(e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)})A(s)v_0
\end{aligned}
$$

one deduces

$$
\|[\Phi_1(t,s) - \Phi_1(\tau,s)]v_0\| \le C\|v_0\|_{D_0}\Big\{(t-\tau)^\rho(t-s)^{\beta-1} + \frac{t-\tau}{t-s}(\tau-s)^{\rho+\beta-1}\Big\}. \quad (2.20)
$$

By the way, it is possible to prove also that the solution Φ to equation (3.2) is Hölder continuous in its first variable as well other properties of use in the sequel only. Such properties are stated in the following Lemmata 2.1 and 2.2, whose proofs are postponed to the end of this Section, not to break our discussion.

LEMMA 2.1 *For any $\rho \in (1-\beta, 1)$, $\nu \in (0, \rho+\beta-1)$ and $0 \le s < t_1 < t_2 \le T_0$ function Φ satisfies the estimate*

$$
\|\Phi(t_2,s) - \Phi(t_1,s)\| \le C(T_0)(t_2-t_1)^\nu(t_1-s)^{\rho+\beta-2-\nu}. \qquad (2.21)
$$

LEMMA 2.2 *For any $\rho \in (1-\beta, 1)$, $\nu \in (0, \rho+\beta-1)$ function Φ satisfies* *the estimates*

$$\|\Phi(t,s)A(s)^{-1}\| \le C(T_0)(t-s)^{\rho+\beta-1}, \quad 0 \le s < t \le T_0, \tag{2.22}$$

$$\|[\Phi(t_2,s)-\Phi(t_1,s)]A(s)^{-1}\| \le C(T_0)(t_2-t_1)^{\nu}(t_2-s)^{\rho+\beta-1-\nu}, \quad 0 \le s < t_1 < t_2 \le T_0. \tag{2.23}$$

Going on, note that for $0 \le s < t-\varepsilon < t$ one gets

$$\begin{aligned} D_t \int_s^{t-\varepsilon} e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau &= e^{-\varepsilon A(t-\varepsilon)}\Phi_1(t-\varepsilon,s) \\ &- \int_s^{t-\varepsilon} A(\tau)e^{-(t-\tau)A(\tau)}\left(\Phi_1(\tau,s)-\Phi_1(t,s)\right)d\tau + \int_s^{t-\varepsilon} G(t,\tau)d\tau\Phi_1(t,s) \\ &- (e^{-\varepsilon A(t)} - e^{-(t-s)A(t)})\Phi_1(t,s) \\ &= e^{-\varepsilon A(t-\varepsilon)}(\Phi_1(t-\varepsilon,s)-\Phi_1(t,s)) + (e^{-\varepsilon A(t-\varepsilon)} - e^{-\varepsilon A(t)})\Phi_1(t,s) \\ &+ e^{-(t-s)A(t)}\Phi_1(t,s) - \int_s^{t-\varepsilon} A(\tau)e^{-(t-\tau)A(\tau)}\left(\Phi_1(\tau,s)-\Phi_1(t,s)\right)d\tau \\ &+ \int_s^{t-\varepsilon} G(t,\tau)d\tau\Phi_1(t,s). \end{aligned} \tag{2.24}$$

By virtue of (2.24) together with *H11*, (2.8), (2.19), (2.16) and

$$\|e^{-\varepsilon A(t-\varepsilon)} - e^{-\varepsilon A(t)}\| \le C\varepsilon^{\rho+2\beta-2}, \tag{2.25}$$

which follows from (2.17), $\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau$ is differentiable in $\mathcal{L}(L^p(\Omega))$ with respect to $t \in (s,T_0]$ and

$$\begin{aligned} D_t \int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau &= e^{-(t-s)A(t)}\Phi_1(t,s) \\ &- \int_s^t A(\tau)e^{-(t-\tau)A(\tau)}\left(\Phi_1(\tau,s)-\Phi_1(t,s)\right)d\tau + \int_s^t G(t,\tau)d\tau\Phi_1(t,s). \end{aligned} \tag{2.26}$$

Estimating the norm of each term of the right-hand side in (2.26) one obtains

$$\begin{aligned} \left\|D_t \int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau\right\| &\le C(t-s)^{\beta-1}(t-s)^{\rho+\beta-2} \\ &+ C\int_s^t (t-\tau)^{\beta-2}\left\{(t-\tau)^{\rho}(t-s)^{\beta-2} + \frac{t-\tau}{t-s}(\tau-s)^{\rho+\beta-2}\right\}d\tau \\ &+ C\int_s^t (t-\tau)^{\rho+2\beta-3}d\tau(t-s)^{\rho+\beta-2} \le C\left\{(t-s)^{\rho+2\beta-3} + (t-s)^{2\rho+3\beta-4}\right\} \\ &\le C(t-s)^{\rho+2\beta-3}, \quad 0 \le s < t \le T_0. \end{aligned} \tag{2.27}$$

Set now

$$W(t,s) = \int_s^t e^{-(t-\tau)A(\tau)}\Phi(\tau,s)d\tau. \tag{2.28}$$

Then

$$U(t,s) = e^{-(t-s)A(s)} + W(t,s), \tag{2.29}$$

and, in view of (2.8) and (2.12), one gets

$$\|W(t,s)\| \le C(t-s)^{\rho+2\beta-2}, \quad 0 \le s < t \le T_0. \tag{2.30}$$

By virtue of (2.6) one has

$$W(t,s) = -\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau - \int_s^t \int_\sigma^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,\sigma)d\tau\, \Phi(\sigma,s)d\sigma. \tag{2.31}$$

Hence, in view of *H11*, (2.13) and (2.27) $W(t,s)$ is differentiable with respect to $t \in (s, T_0]$ and

$$\begin{aligned} D_t W(t,s) = &-D_t \int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau \\ &- \int_s^t D_t \int_\sigma^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,\sigma)d\tau\, \Phi(\sigma,s)d\sigma. \end{aligned} \tag{2.32}$$

As a consequence

$$\|D_t W(t,s)\| \le C(t-s)^{\rho+2\beta-3}, \quad 0 \le s < t \le T_0. \tag{2.33}$$

From (2.8), (2.30) and (2.33) it follows that

$$\|U(t,s)\| \le C(t-s)^{\beta-1}, \quad \|D_t U(t,s)\| \le C(t-s)^{\beta-2}, \quad 0 \le s < t \le T_0. \tag{2.34}$$

Then, if we assume that $v_0 \in D_0$, we can show that

$$\|D_t W(t,s)v_0\| \le C(t-s)^{\rho+2\beta-2}\|v_0\|_{D_0}, \quad v_0 \in D_0,\ 0 \le s < t \le T_0. \tag{2.35}$$

For this purpose it suffices to take estimates (2.20), (2.22), (2.23), with $\nu + \beta - 1 > 0$, into account and to repeat the same procedure followed to deduce estimate (2.33).

LEMMA 2.3 *For $f \in L^p(\Omega)$ one has $\int_s^t e^{-(t-\tau)A(t)} f d\tau \in D(A(t))$ for all $t \in [0, T_0]$ and $s \in [0, t)$ and the following formula holds:*

$$A(t)\int_s^t e^{-(t-\tau)A(t)} f d\tau = \{I - e^{-(t-s)A(t)}\}f, \quad 0 \le s < t \le T_0. \tag{2.36}$$

Proof. We follow the idea of E. Sinestrari [3; Proposition 1.2]. Let λ be an element in the resolvent set $\rho(A(t))$ of $A(t)$. Then, for $0 < \varepsilon < t - s$, we have

$$\begin{aligned} &A(t)(\lambda + A(t))^{-1} \int_s^{t-\varepsilon} e^{-(t-\tau)A(t)} f d\tau \\ &= (\lambda + A(t))^{-1} \int_s^{t-\varepsilon} A(t) e^{-(t-\tau)A(t)} f d\tau \\ &= (\lambda + A(t))^{-1} \int_s^{t-\varepsilon} \frac{\partial}{\partial \tau} e^{-(t-\tau)A(t)} f d\tau \\ &= (\lambda + A(t))^{-1} \{e^{-\varepsilon A(t)} - e^{-(t-s)A(t)}\} f \\ &= e^{-\varepsilon A(t)} (\lambda + A(t))^{-1} f - (\lambda + A(t))^{-1} e^{-(t-s)A(t)} f. \end{aligned}$$

Letting $\varepsilon \to 0+$, we get

$$\begin{aligned} &\lambda(\lambda + A(t))^{-1} \int_s^t e^{-(t-\tau)A(t)} f d\tau - \int_s^t e^{-(t-\tau)A(t)} f d\tau \\ &= A(t)(\lambda + A(t))^{-1} \int_s^t e^{-(t-\tau)A(t)} f d\tau \\ &= (\lambda + A(t))^{-1} f - (\lambda + A(t))^{-1} e^{-(t-s)A(t)} f. \end{aligned}$$

This shows that $\int_s^t e^{-(t-\tau)A(t)} f d\tau \in D(A(t))$, $0 \le s < t \le T_0$, and (2.36) holds. $\square$

As is easily seen for $0 < \varepsilon < t - s$

$$\begin{aligned} &A(t) \int_s^{t-\varepsilon} e^{-(t-\tau)A(\tau)} \Phi_1(\tau, s) d\tau \\ &= \int_s^{t-\varepsilon} \Phi_1(t, \tau) \Phi_1(\tau, s) d\tau + \int_s^{t-\varepsilon} A(\tau) e^{-(t-\tau)A(\tau)} (\Phi_1(\tau, s) - \Phi_1(t, s)) d\tau \\ &\quad - \int_s^{t-\varepsilon} G(t, \tau) \Phi_1(t, s) d\tau + A(t) \int_s^{t-\varepsilon} e^{-(t-\tau)A(t)} \Phi_1(t, s) d\tau. \end{aligned} \tag{2.37}$$

Hence

$$\begin{aligned} &\int_s^{t-\varepsilon} e^{-(t-\tau)A(\tau)} \Phi_1(\tau, s) d\tau = A(t)^{-1} \int_s^{t-\varepsilon} \Phi_1(t, \tau) \Phi_1(\tau, s) d\tau \\ &+ A(t)^{-1} \int_s^{t-\varepsilon} A(\tau) e^{-(t-\tau)A(\tau)} (\Phi_1(\tau, s) - \Phi_1(t, s)) d\tau \\ &- A(t)^{-1} \int_s^{t-\varepsilon} G(t, \tau) \Phi_1(t, s) d\tau + \int_s^{t-\varepsilon} e^{-(t-\tau)A(t)} \Phi_1(t, s) d\tau. \end{aligned}$$

Letting $\varepsilon \to 0+$, one deduces

$$\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau = A(t)^{-1}\int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau$$
$$+A(t)^{-1}\int_s^t A(\tau)e^{-(t-\tau)A(\tau)}(\Phi_1(\tau,s)-\Phi_1(t,s))d\tau$$
$$-A(t)^{-1}\int_s^t G(t,\tau)\Phi_1(t,s)d\tau + \int_s^t e^{-(t-\tau)A(t)}\Phi_1(t,s)d\tau, \quad 0 \le s < t \le T_0.$$

This equality and Lemma 2.3 imply that $\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau \in D(A(t))$ for all $t \in [s, T_0]$ and

$$A(t)\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau$$
$$= \int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau + \int_s^t A(\tau)e^{-(t-\tau)A(\tau)}(\Phi_1(\tau,s)-\Phi_1(t,s))d\tau$$
$$-\int_s^t G(t,\tau)\Phi_1(t,s)d\tau + \{I - e^{-(t-s)A(t)}\}\Phi_1(t,s). \tag{2.38}$$

Similarly to (2.27) one gets

$$\left\| A(t)\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau \right\| \le C(t-s)^{\rho+2\beta-3}, \quad 0 \le s < t \le T_0. \tag{2.39}$$

It follows from (2.31) and (2.39) that

$$\|A(t)W(t,s)\| \le C(t-s)^{\rho+2\beta-3}, \quad 0 \le s < t \le T_0. \tag{2.40}$$

From (2.26) and (2.38) it follows that

$$D_t\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau + A(t)\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau$$
$$= \Phi_1(t,s) + \int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau, \quad 0 \le s < t \le T_0. \tag{2.41}$$

With the aid of (2.31), (2.32), (2.41) and (2.6) we obtain

$$D_tW(t,s) + A(t)W(t,s) = -D_t\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau$$
$$-\int_s^t D_t\int_\sigma^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,\sigma)d\tau\Phi(\sigma,s)d\sigma$$
$$-A(t)\int_s^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,s)d\tau$$
$$-\int_s^t A(t)\int_\sigma^t e^{-(t-\tau)A(\tau)}\Phi_1(\tau,\sigma)d\tau\Phi(\sigma,s)d\sigma$$

$$= -\Phi_1(t,s) - \int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau$$

$$- \int_s^t \Big\{\Phi_1(t,\sigma) + \int_\sigma^t \Phi_1(t,\tau)\Phi_1(\tau,\sigma)d\tau\Big\}\Phi(\sigma,s)d\sigma$$

$$= -\Phi_1(t,s) - \int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau - \int_s^t \Phi_1(t,\sigma)\Phi(\sigma,s)d\sigma$$

$$- \int_s^t \Phi_1(t,\tau)\int_s^\tau \Phi_1(\tau,\sigma)\Phi(\sigma,s)d\sigma d\tau$$

$$= -\Phi_1(t,s) - \int_s^t \Phi_1(t,\tau)\Phi_1(\tau,s)d\tau - \int_s^t \Phi_1(t,\sigma)\Phi(\sigma,s)d\sigma$$

$$- \int_s^t \Phi_1(t,\tau)\{-\Phi(\tau,s) - \Phi_1(\tau,s)\}d\tau = -\Phi_1(t,s), \quad 0 \le s < t \le T_0. \quad (2.42)$$

Using (2.7), (2.29) and (2.42) we easily conclude

$$D_tU(t,s) + A(t)U(t,s) = 0, \quad 0 \le s < t \le T_0. \quad (2.43)$$

Moreover, the following inequality follows from (2.34) and (2.43):

$$\|A(t)U(t,s)\| \le C(t-s)^{\beta-2}, \qquad 0 \le s < t \le T_0. \quad (2.44)$$

We summarize the properties of operator function U in the following Theorem.

THEOREM 2.1 *Under assumptions H2, H10, H11 and* $\rho \in (1-\beta, 1)$ *operator* U *satisfies estimates (2.34), (2.44) and solves equation (2.43).*

We prove now Lemmata 2.1 and 2.2.

Proof of Lemma 2.1. From equation (2.6) we easily deduce the following identity, where $0 \le s < t_1 < t_2 \le T_0$:

$$\Phi(t_2,s) - \Phi(t_1,s) = -[\Phi_1(t_2,s) - \Phi_1(t_1,s)] - \int_{t_1}^{t_2} \Phi_1(t_2,\tau)\Phi(\tau,s)d\tau$$

$$- \int_s^{t_1} [\Phi_1(t_2,\tau) - \Phi_1(t_1,\tau)]\Phi(\tau,s)d\tau =: \sum_{j=1}^3 \widetilde{\Phi}_j(t_2,t_1,s). \quad (2.45)$$

Then we rewrite estimate (2.19) in the more convenient form

$$\|\Phi_1(t_2,s) - \Phi_1(t_1,s)\| \le C\Big\{(t_2-t_1)^\rho(t_2-s)^{\beta-2} + \frac{t_2-t_1}{t_2-s}(t_1-s)^{\rho+\beta-2}\Big\}$$

$$\le C\Big\{(t_2-t_1)^\nu(t_2-t_1)^{\rho-\nu}(t_2-s)^{\beta-2} + \Big(\frac{t_2-t_1}{t_2-s}\Big)^\nu(t_1-s)^{\rho+\beta-2}\Big\}$$

$$\leq C(t_2-t_1)^\nu\Big\{(t_2-s)^{\rho+\beta-2-\nu}+(t_1-s)^{\rho+\beta-2-\nu}\Big\}$$
$$\leq C(t_2-t_1)^\nu(t_1-s)^{\rho+\beta-2-\nu}, \qquad \forall\nu\in(0,\rho]. \tag{2.46}$$

We conclude the proof by observing that estimate (2.21) is implied by (2.46) and the following inequalities, since $\rho\in(1-\beta,1)$ and $\nu\in(0,\rho+\beta-1)$:

$$\begin{aligned}\|\widetilde{\Phi}_2(t_2,t_1,s)\| &\leq C\int_{t_1}^{t_2}(t_2-\tau)^\nu(t_2-\tau)^{\rho+\beta-2-\nu}(\tau-s)^{\rho+\beta-2}\,d\tau\\ &\leq C(t_2-t_1)^\nu\int_s^{t_2}(t_2-\tau)^{\rho+\beta-2-\nu}(\tau-s)^{\rho+\beta-2}\,d\tau\\ &\leq C(t_2-t_1)^\nu(t_2-s)^{2\rho+2\beta-3-\nu},\\ \|\widetilde{\Phi}_3(t_2,t_1,s)\| &\leq C(t_2-t_1)^\nu\int_s^{t_1}(t_1-\tau)^{\rho+\beta-2-\nu}(\tau-s)^{\rho+\beta-2}\,d\tau\\ &\leq C(t_2-t_1)^\nu(t_1-s)^{2\rho+2\beta-3-\nu}. \ \square\end{aligned}$$

Proof of Lemma 2.2. First we notice that estimate (2.22) easily follows from the estimate

$$\|\Phi_1(t,s)A(s)^{-1}\|\leq C(t-s)^{\rho+\beta-1}, \tag{2.47}$$

and the equation

$$\Phi(t,s)=-\Phi_1(t,s)-\int_s^t\Phi(t,\tau)\Phi_1(\tau,s)d\tau,$$

that is a simple consequence of the integral equation (2.6) defining Φ, which makes use of the identity

$$\int_s^t\Phi(t,\tau)\Phi_1(\tau,s)d\tau=\int_s^t\Phi_1(t,\tau)\Phi(\tau,s)d\tau.$$

Then from the definition (2.12) of Φ_1 and the identity

$$\begin{aligned}&[\Phi_1(t,s)-\Phi_1(\tau,s)]A(s)^{-1}\\ &=(A(t)-A(\tau))A(s)^{-1}e^{-(t-s)A(s)}\\ &\quad+(A(\tau)-A(s))A(s)^{-1}(e^{-(t-s)A(s)}-e^{-(\tau-s)A(s)}),\end{aligned}$$

and the inequality

$$\|e^{-(t-s)A(s)} - e^{-(\tau-s)A(s)}\| = \left\| \int_\tau^t D_r e^{-(r-s)A(s)} dr \right\| \le C \int_\tau^t (r-s)^{\beta-2} dr$$
$$\le C\Big\{(\tau-s)^{\beta-1} - (t-s)^{\beta-1}\Big\} = C(\tau-s)^{\beta-1}\Big\{1 - \Big(\frac{\tau-s}{t-s}\Big)^{1-\beta}\Big\}$$
$$\le C(\tau-s)^{\beta-1}\Big(1 - \frac{\tau-s}{t-s}\Big) = C(\tau-s)^{\beta-1}\frac{t-\tau}{t-s}$$
$$\le C(\tau-s)^{\beta-1}\Big(\frac{t-\tau}{t-s}\Big)^{\nu} \le C(t-\tau)^{\nu}(\tau-s)^{\beta-1-\nu}, \quad \forall \nu \in (0,1],$$

we easily deduce (cf. (2.10)) the following estimates, where $\rho \in (1-\beta, 1)$ and $\nu \in (0, \rho+\beta-1)$:

$$\|[\Phi_1(t_2,s) - \Phi_1(t_1,s)]A(s)^{-1}\|$$
$$\le C(t_2-t_1)^{\rho}(t_2-s)^{\beta-1} + C(t_2-t_1)^{\nu}(t_1-s)^{\rho+\beta-1-\nu}$$
$$\le C(t_2-t_1)^{\nu}(t_2-s)^{\rho+\beta-1-\nu}. \tag{2.48}$$

Observe now that estimate (2.23) is a consequence of (2.22), (2.48) and the following identity (cf. (2.6)), where $0 \le s \le t_1 < t_2 \le T$:

$$\Phi(t_2,s) - \Phi(t_1,s) = -[\Phi_1(t_2,s) - \Phi_1(t_1,s)]$$
$$- \int_{t_1}^{t_2} \Phi_1(t_2,\tau)\Phi(\tau,s)d\tau - \int_s^{t_1} [\Phi_1(t_2,\tau) - \Phi_1(t_1,\tau)]\Phi(\tau,s)d\tau. \ \square \tag{2.49}$$

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3 Existence of the solution to the Cauchy problem (2.1)

Let $f \in C^{\rho}([0,T_0]; L^p(\Omega))$, $\rho \in (1-\beta, 1)$. The argument by which we derived (2.26) yields that $t \to \int_0^t e^{-(t-s)A(s)} f(s)ds$ is differentiable in $(0, T_0]$ and

$$D_t \int_0^t e^{-(t-s)A(s)} f(s)ds = e^{-tA(t)} f(t)$$
$$- \int_0^t A(s)e^{-(t-s)A(s)}[f(s) - f(t)]ds + \int_0^t G(t,s)f(t)ds, \quad t \in (0, T_0].$$

In view of *H11*, (2.30) and (2.33) $t \to \int_0^t W(t,s)f(s)ds$ is differentiable in $(0, T_0]$ and

$$D_t \int_0^t W(t,s)f(s)ds = \int_0^t D_t W(t,s)f(s)ds, \quad t \in (0, T_0].$$

Hence $t \to \int_0^t U(t,s)f(s)ds$ is differentiable in $(0,T_0]$, and

$$D_t \int_0^t U(t,s)f(s)ds = e^{-tA(t)}f(t)$$
$$- \int_0^t A(s)e^{-(t-s)A(s)}[f(s) - f(t)]ds$$
$$+ \int_0^t G(t,s)f(t)ds + \int_0^t D_tW(t,s)f(s)ds, \quad t \in (0,T_0]. \tag{3.1}$$

For $0 < \varepsilon < t - s$ we have

$$A(t)\int_0^{t-\varepsilon} e^{-(t-s)A(s)}f(s)ds = \int_0^{t-\varepsilon} A(t)e^{-(t-s)A(s)}f(s)ds$$
$$= \int_0^{t-\varepsilon} \Phi_1(t,s)f(s)ds + \int_0^{t-\varepsilon} A(s)e^{-(t-s)A(s)}(f(s) - f(t))ds$$
$$- \int_0^{t-\varepsilon} G(t,s)f(t)ds + \int_0^{t-\varepsilon} A(t)e^{-(t-s)A(t)}f(t)ds. \tag{3.2}$$

With the aid of the argument by which we derived (2.38) from (2.37) and (3.2) we obtain that $\int_0^t e^{-(t-s)A(t)}f(t)ds \in D(A(t))$ for all $t \in (0,T_0]$ and

$$A(t)\int_0^t e^{-(t-s)A(s)}f(s)ds = \int_0^t \Phi_1(t,s)f(s)ds$$
$$+ \int_0^t A(s)e^{-(t-s)A(s)}(f(s) - f(t))ds - \int_0^t G(t,s)f(t)ds + \{I - e^{-tA(t)}\}f(t).$$

In view of (2.40) we see that $\int_0^t W(t,s)f(s)ds \in D(A(t))$ for all $t \in (0,T_0]$ and

$$A(t)\int_0^t W(t,s)f(s)ds = \int_0^t A(t)W(t,s)f(s)ds.$$

Hence one observes that $\int_0^t U(t,s)f(s)ds \in D(A(t))$ for all $t \in (0,T_0]$ and

$$A(t)\int_0^t U(t,s)f(s)ds$$
$$= \int_0^t \Phi_1(t,s)f(s)ds + \int_0^t A(s)e^{-(t-s)A(s)}[f(s) - f(t)]ds$$
$$- \int_0^t G(t,s)f(t)ds + \{I - e^{-tA(t)}\}f(t) + \int_0^t A(t)W(t,s)f(s)ds. \tag{3.3}$$

From (2.42), (3.1) and (3.3) we conclude that

$$D_t \int_0^t U(t,s)f(s)ds + A(t)\int_0^t U(t,s)f(s)\,ds = f(t) + \int_0^t \Phi_1(t,s)f(s)\,ds$$

$$+ \int_0^t \{D_t W(t,s) + A(t)W(t,s)\}\, f(s)ds = f(t), \quad t \in (0,T_0]. \tag{3.4}$$

By virtue of (2.43) and (3.4) we have established the following result.

THEOREM 3.1 *Let $v_0 \in L^p(\Omega)$ and $f \in C^\rho([0,T_0];L^p(\Omega))$, $\rho \in (1-\beta,1)$. Then, under assumptions H2, H10, H11, the function v defined by*

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)ds, \qquad t \in (0,T_0], \tag{3.5}$$

is differentiable in $L^p(\Omega)$ in $(0,T_0]$, belongs to $D(A(t))$ for $0 < t \le T_0$ and satisfies the equation

$$D_t v(t) + A(t)v(t) = f(t), \quad 0 < t \le T_0. \tag{3.6}$$

In order to obtain a sufficient condition for the initial condition $v(0) = v_0$ to hold, we prepare the following Lemma 3.1, where the interpolation spaces $(D_0,X)_{\theta,p}$, $0<\theta<1$, $1\le p\le\infty$, are defined by (1.14).

LEMMA 3.1 *Let $\{e^{-tA}\}_{t>0}$ be an infinitely differentiable semigroup in a Banach space X satisfying an estimate of type (2.2) for some $\beta \in (0,1)$ in a sector $\Sigma \subset \mathbf{C}$ with half-width $\varphi \in (\pi/2,\pi)$. Then the following estimate holds for all $v \in (D_0,X)_{\theta,p}$, where $D_0 = D(A)$ and $0<\theta<1$, $1\le p\le\infty$:*

$$\|e^{-tA}v - v\|_X \le Ct^{\beta-\theta}\|v\|_{(D_0,X)_{\theta,p}}, \tag{3.7}$$

$$\left\|D_t^j e^{-tA}v\right\|_X \le Ct^{\beta-j-\theta}\|v\|_{(D_0,X)_{\theta,p}}, \qquad j \in \mathbf{N}\backslash\{0\}. \tag{3.8}$$

Proof. For $v \in X$

$$\|e^{-tA}v - v\|_X \le \|e^{-tA}v\|_X + \|v\|_X \le Ct^{\beta-1}\|v\|_X. \tag{3.9}$$

Let $v \in D_0$. Since $\lim_{t\to 0} e^{-tA}v = v$ then, one has for $w \in Av$

$$\|e^{-tA}v - v\|_X = \left\|\int_0^t D_s e^{-sA}v ds\right\|_X = \left\|\int_0^t D_s e^{-sA}A^{-1}w ds\right\|_X$$

$$= \left\|\int_0^t e^{-sA}w ds\right\|_X \le C\int_0^t s^{\beta-1}\|w\|_X ds \le Ct^\beta\|w\|_X.$$

Hence

$$\|e^{-tA}v - v\|_X \le Ct^\beta\|v\|_{D_0}. \tag{3.10}$$

The inequality (3.7) follows from (3.9) and (3.10). Analogously (3.8) is a consequence of

$$\left\|D_t^j e^{-tA}v\right\|_X \leq \begin{cases} Ct^{\beta-j-1}\|v\|_X, \\ Ct^{\beta-j}\|v\|_{D_0}, \end{cases} \qquad j \in \mathbf{N}\backslash\{0\}. \ \square$$

We can now exhibit a large enough subspace in $(D_0, L^p(\Omega))_{\theta,p}$.

LEMMA 3.2 *The following inclusion holds true for* $\theta \in (0,1)\backslash\{1/(2p), 1/2\}$:

$$(D_0, L^p(\Omega))_{\theta,p} \supset \mathcal{W}_0^{2\theta,p}(\Omega) = \left\{v \in L^p(\Omega) : (v/m_0) \in W_0^{2\theta,p}(\Omega)\right\}, \tag{3.11}$$

where

$$W_0^{2\theta,p}(\Omega) = \begin{cases} W^{2\theta,p}(\Omega), & \theta \in (0, 1/(2p)), \\ \{u \in W^{2\theta,p}(\Omega) : u = 0 \ \ on\ \partial\Omega\}, & \theta \in (1/(2p), 1). \end{cases} \tag{3.12}$$

Proof. First we recall (cf. *H1*) that

$$D_0 = M(0)D(L(0)),\ A(0)v = L(0)[m_0^{-1}v],\ M(0)v(x) = m_0(x)v(x), \tag{3.13}$$

$m_0 \in C(\overline{\Omega})$ being the a.e. positive function in $\overline{\Omega}$ introduced in assumption *H1*.

We endow D_0 and $D(L(0))$ with their own graph-norms, i.e.,

$$\|v\|_{D_0} = \|v\|_{L^p(\Omega)} + \|L(0)(v/m_0)\|_{L^p(\Omega)},$$

$$\|u\|_{D(L(0))} = \|u\|_{L^p(\Omega)} + \|L(0)u\|_{L^p(\Omega)}.$$

Observe now that $M(0)$ continuously maps $L^p(\Omega)$ and $D(L(0))$ into $L^p(\Omega)$ and D_0, respectively, and satisfies the estimates

$$\|M(0)v\|_{L^p(\Omega)} \leq \|m_0\|_{C(\overline{\Omega})}\|u\|_{L^p(\Omega)}, \tag{3.14}$$

$$\begin{aligned} \|M(0)v\|_{D_0} &= \|M(0)v\|_{L^p(\Omega)} + \|L(0)v\|_{L^p(\Omega)} \\ &\leq \max\{\|m_0\|_{C(\overline{\Omega})}, 1\}\|v\|_{D(L(0))}. \end{aligned} \tag{3.15}$$

Hence we deduce that $M(0) \in \mathcal{L}((D(L(0)); L^p(\Omega))_{\theta,p}; (D_0; L^p(\Omega))_{\theta,p})$. In particular (cf. [9]), for $\theta \neq 1/(2p)$, we have proved the inclusion

$$M(0)W_0^{2\theta,p}(\Omega) = M(0)(D(L(0)); L^p(\Omega))_{\theta,p} \subset (D_0; L^p(\Omega))_{\theta,p}. \tag{3.16}$$

Finally, it is easy to check that $M(0)W_0^{2\theta,p}(\Omega)$ coincides with the vector space defined in the right-hand side in (3.11). $\square$

We now go on with our discussion. Taking $\theta = \beta$ in (3.7) and (3.8), one obtains

$$\|e^{-tA}v - v\|_X \leq C\|v\|_{(D_0,X)_{\beta,p}}, \tag{3.17}$$

$$\left\|D_t e^{-tA}v\right\|_X \leq Ct^{-1}\|v\|_{(D_0,X)_{\beta,p}}. \tag{3.18}$$

Applying (3.17) and (3.18) to the present case: $A = A(0) = L(0)M(0)^{-1}$, $X = L^p(\Omega)$, one has

$$\|e^{-tA(0)}v - v\|_{L^p(\Omega)} \leq C\|v\|_{(D_0, L^p(\Omega))_{\beta,p}}, \tag{3.19}$$

$$\left\|D_t e^{-tA(0)}v\right\|_{L^p(\Omega)} \leq Ct^{-1}\|v\|_{(D_0, L^p(\Omega))_{\beta,p}}. \tag{3.20}$$

In view of Théorème 2.1 on p. 22 in [5], D_0 is dense in $(D_0, L^p(\Omega))_{\beta,p}$. This property and (3.19) yield that

$$\lim_{t\to 0+} \|e^{-tA(0)}v - v\|_{L^p(\Omega)} = 0 \tag{3.21}$$

for $v \in (D_0, L^p(\Omega))_{\beta,p}$. This, together with *H5* and (2.30), implies

$$U(t,0)v_0 = e^{-tA(0)}v_0 + W(t,0)v_0 \to v_0 \quad \text{as } t \to 0+, \tag{3.22}$$

for each $v_0 \in (D_0, L^p(\Omega))_{\beta,p}$. Consequently, function $t \to U(t,0)v_0 \in C([0,T]; X)$ for all $v_0 \in (D_0, L^p(\Omega))_{\beta,p}$ and satisfies the estimate (cf. (2.30))

$$\begin{aligned} \|U(t,0)v_0\|_{C([0,T];X)} &\leq \|e^{-tA(0)}v_0 - v_0\|_{L^p(\Omega)} + \|v_0\|_{L^p(\Omega)} + \|W(t,0)v_0\|_{L^p(\Omega)} \\ &\leq \|v_0\|_{(D_0,L^p(\Omega))} + \|v_0\|_{L^p(\Omega)} + C(T_0)t^{\rho+2\beta-2}\|v_0\|_{L^p(\Omega)}, \quad t \in [0,T_0]. \end{aligned} \tag{3.23}$$

Moreover, function $t \to \int_0^t U(t,s)f(s)ds$ belongs to $C([0,T];X)$ since it is continuously differentiable in $(0,T_0]$ and tends to 0 as $t \to 0+$ according to estimate (2.30). It, in turn, implies the estimate

$$\left\|\int_0^t U(t,s)f(s)ds\right\| \leq C(T_0)t^\beta \|f\|_{C([0,T];X)}, \quad t \in [0,T_0]. \tag{3.24}$$

Thus the following theorem has been established.

THEOREM 3.2 *Let $v_0 \in (D_0, L^p(\Omega))_{\beta,p}$ and $f \in C^\rho([0,T_0]; L^p(\Omega))$, $\rho \in (1-\beta, 1)$. Then, under assumptions H2, H10, H11, the function v defined by*

$$v(t) = U(t,0)v_0 + \int_0^t U(t,s)f(s)ds, \qquad t \in [0,T_0], \tag{3.25}$$

is a solution to the initial value problem

$$\begin{aligned} &D_t v(t) + A(t)v(t) = f(t), \quad 0 < t \leq T_0, \\ &v(0) = v_0, \end{aligned} \tag{3.26}$$

and satisfies the estimate

$$\|v(t)\| \leq C(T_0)\left[\|v_0\|_{(D_0,L^p(\Omega))} + t^\beta\|f\|_{C([0,T];X)}\right], \quad t \in [0,T_0]. \tag{3.27}$$

Further, function $w(t) = M(t)^{-1}v(t)$ solves the initial value problem

$$\begin{aligned} &D_t[M(t)w(t)] + L(t)w(t) = f(t), \quad 0 < t \leq T_0, \\ &M(0)w(0) = v_0. \end{aligned} \tag{3.28}$$

4 Uniqueness of the solution to the Cauchy problem (2.1)

To prove the uniqueness of the solution we consider problem (2.1) in the space $H^{-1}(\Omega)$. The restriction of $\mathcal{L}(t)$ to $H_0^1(\Omega)$ is denoted by $\tilde{L}(t)$. Then, $\tilde{A}(t) = \tilde{L}(t)M(t)^{-1}$ is a single-valued linear operator in $H^{-1}(\Omega)$, and condition (P) in [3], p. 80, is satisfied with $\alpha = \beta = 1$. Clearly, according to assumption *H1*, $D(\tilde{A}(t)) = \{m(\cdot,t)u : u \in H_0^1(\Omega)\} = \{m_0 u : u \in H_0^1(\Omega)\}$ is independent of $t \in [0, T_0]$.

One can also show that $D(\tilde{A}(t))$ is dense in $H^{-1}(\Omega)$ replacing both $L^{p'}(\Omega)$ and $D(L(t))$ with $H_0^1(\Omega)$ in the proof of the density of $D(A(t))$ in $L^p(\Omega)$. It is easily verified that

$$\|(\tilde{A}(t) - \tilde{A}(s))\tilde{A}(0)^{-1}\|_{\mathcal{L}(H^{-1}(\Omega))} \le C|t-s|^\rho, \quad s,t \in [0, T_0].$$

Therefore, using a classical result one can construct the fundamental solution $\tilde{U}(t,s)$ to the problem

$$\begin{aligned} &D_t v(t) + \tilde{A}(t)v(t) = f(t), \quad 0 < t \le T_0, \\ &v(0) = v_0, \end{aligned} \tag{4.1}$$

in $H^{-1}(\Omega)$ as follows:

$$\tilde{U}(t,s) = e^{-(t-s)\tilde{A}(s)} + \int_s^t e^{-(t-\tau)\tilde{A}(\tau)}\tilde{\Phi}(\tau,s)d\tau, \quad 0 \le s < t \le T_0,$$

where $\tilde{\Phi}$ solves the Volterra integral equation

$$\tilde{\Phi}(t,s) + \tilde{\Phi}_1(t,s) + \int_s^t \tilde{\Phi}_1(t,\tau)\tilde{\Phi}(\tau,s)d\tau = 0,$$

with

$$\tilde{\Phi}_1(t,s) = (\tilde{A}(t) - \tilde{A}(s))e^{-(t-s)\tilde{A}(s)}.$$

LEMMA 4.1 *Let $q \ge 2n/(n+2)$ and $q \ge p' = p/(p-1)$. Suppose that $u \in D(L(t))$ and $\tilde{u} \in H_0^1(\Omega)$. Then for any $w \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$*

$$(u, \mathcal{L}(t)w) = (L(t)u, w) \quad \text{and} \quad (\tilde{u}, \mathcal{L}(t)w) = (\tilde{L}(t)\tilde{u}, w).$$

Proof. By virtue of well-known imbedding theorems we have $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow H_0^1(\Omega) \hookrightarrow L^{q'}(\Omega)$. Let $w \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$. Since $q \ge p'$, $w \in W^{2,p'}(\Omega) \cap W_0^{1,p'}(\Omega) = D(L(t)^*)$. Therefore $(u, \mathcal{L}(t)w) = (u, L(t)^*w) = (L(t)u, w)$. Let $\{\varphi_k\}$ be a sequence of elements of $C_0^\infty(\Omega)$ such that $\varphi_k \to \tilde{u}$ in $H_0^1(\Omega) \hookrightarrow L^{q'}(\Omega)$. Then, since $D_{x_j}w \in W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, one observes

$$-\int_\Omega \tilde{u} D_{x_i}(a_{i,j}D_{x_j}\overline{w})dx = -\lim_{k\to+\infty}\int_\Omega \varphi_k D_{x_i}(a_{i,j}D_{x_j}\overline{w})dx$$
$$= \lim_{k\to+\infty}\int_\Omega a_{i,j}D_{x_i}\varphi_k D_{x_j}\overline{w}dx = \int_\Omega a_{i,j}D_{x_i}\tilde{u}D_{x_j}\overline{w}dx.$$

Therefore

$$(\tilde{u}, \mathcal{L}(t)w) = \int_\Omega \tilde{u}\Big\{-\sum_{i,j=1}^n D_{x_i}(a_{i,j}D_{x_j}\overline{w}) + a\overline{w}\Big\}dx$$
$$= \int_\Omega \Big\{\sum_{i,j=1}^n a_{i,j}D_{x_i}\tilde{u}D_{x_j}\overline{w} + au\overline{w}\Big\}dx = (\tilde{L}(t)\tilde{u}, w). \square$$

REMARK 4.1 If $p \geq 2n/(n+2)$, the statement of Lemma 4.1 is evident, since $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow H_0^1(\Omega)$.

COROLLARY 4.1 *$L(t)u = \tilde{L}(t)u$ for $u \in D(L(t)) \cap H_0^1(\Omega)$, and $A(t)v = \tilde{A}(t)v$ for $v \in D(A(t)) \cap D(\tilde{A}(t))$.*

LEMMA 4.2 *Let $f \in L^p(\Omega) \cap H^{-1}(\Omega)$. Then, $(\lambda M + L(t))^{-1}f = (\lambda M + \tilde{L}(t))^{-1}f$ for each $0 \leq t \leq T_0$.*

Proof. Set $u = (\lambda M + L(t))^{-1}f$ and $\tilde{u} + (\lambda M + \tilde{L}(t))^{-1}f$. Let g be an arbitrary element of $L^q(\Omega)$, where q is the number in the previous lemma, and let w be the solution to the Dirichlet problem for $g = (\overline{\lambda}m(\cdot,t) + \mathcal{L}(t))w$. Then $w \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$, and by the previous lemma $(u, \mathcal{L}(t)w) = (L(t)u, w)$ and $(\tilde{u}, \mathcal{L}(t)w) = (\tilde{L}(t)\tilde{u}, w)$. Therefore

$$(u, g) = (u, (\overline{\lambda}m(\cdot,t) + \mathcal{L}(t))w) = \lambda(m(\cdot,t)u, w) + (u, \mathcal{L}(t)w)$$
$$= \lambda(m(\cdot,t)u, w) + (L(t)u, w) = (\lambda m(\cdot,t)u + L(t)u, w) = (f, w), \quad (4.2)$$
$$(\tilde{u}, g) = (\tilde{u}, (\overline{\lambda}m(\cdot,t) + \mathcal{L}(t))w) = \lambda(m(\cdot,t)\tilde{u}, w) + (\tilde{u}, \mathcal{L}(t)w)$$
$$= \lambda(m(\cdot,t)\tilde{u}, w) + (\tilde{L}(t)\tilde{u}, w) = (\lambda m(\cdot,t)\tilde{u} + \tilde{L}(t)\tilde{u}, w) = (f, w). \quad (4.3)$$

It follows from (4.2) and (4.3) that $(u, g) = (\tilde{u}, g)$ for any $g \in L^q(\Omega)$, from which the conclusion follows. $\square$

COROLLARY 4.2 *If $f \in L^p(\Omega) \cap H^{-1}(\Omega)$, then $(\lambda + A(t))^{-1}f = (\lambda + \tilde{A}(t))^{-1}f$ and $U(t,s)f = \tilde{U}(t,s)f$.*

Proof. The first statement readily follows from the lemma since $(\lambda + A(t))^{-1}f = M(t)(\lambda M(t)+L(t))^{-1}f$, $(\lambda+\tilde{A}(t))^{-1}f = M(t)(\lambda M(t)+\tilde{L}(t))^{-1}f$. The second assertion is a direct consequence of the first one and the construction of $U(t,s)$ and $\tilde{U}(t,s)$. □

THEOREM 4.1 *For $v \in D_0$, $U(t,s)v$ is differentiable with respect to s in the strong topology of $L^p(\Omega)$ and*

$$D_s U(t,s)v = U(t,s)A(s)v. \tag{4.4}$$

Proof. If $v \in D(\tilde{A}(t))$, then by virtue of Theorem 2.1 of Chapter 5 of [5] $\tilde{U}(t,s)v$ is differentiable in s in the strong topology of $H^{-1}(\Omega)$ and (4.4) holds with $U(t,s)$ and $A(s)$ replaced by $\tilde{U}(t,s)$ and $\tilde{A}(s)$ respectively. Hence

$$\tilde{U}(t,s)v - \tilde{U}(t,s')v = \int_{s'}^{s} \tilde{U}(t,\sigma)\tilde{A}(\sigma)v\, d\sigma \tag{4.5}$$

for $0 \le s' < s < t \le T_0$. Suppose that $v \in D(A(t))$. Then $M(t)^{-1}v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Let $q \ge 2n/(n+2)$ and $q \ge p' = p/(p-1)$ as in Lemma 3. Then, $q \ge p' > 2 > p$, $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$ and $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow H_0^1(\Omega)$.

Let $\{u_j\}$ be a sequence in $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ such that $u_j \to M(t)^{-1}v$ in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Then (4.5) holds for $v_j = M(t)u_j$ in place of v, since

$$M(t)^{-1}v_j = u_j \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow H_0^1(\Omega).$$

Hence $v_j \in D(\tilde{A}(t))$ and

$$\tilde{U}(t,s)v_j - \tilde{U}(t,s')v_j = \int_{s'}^{s} \tilde{U}(t,\sigma)\tilde{A}(\sigma)v_j d\sigma. \tag{4.6}$$

Since $u_j \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \hookrightarrow W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = D(L(t))$, one has $v_j = M(t)u_j \in D_0$. Therefore, in view of Lemma 4.1, Corollary 4.1 and Lemma 4.2 we get

$$\tilde{U}(t,s)v_j = U(t,s)v_j,\ \tilde{U}(t,s')v_j = U(t,s')v_j,\ \tilde{U}(t,\sigma)\tilde{A}(\sigma)v_j = U(t,\sigma)A(\sigma)v_j.$$

Substituting this in (4.6) one gets

$$U(t,s)v_j - U(t,s')v_j = \int_{s'}^{s} U(t,\sigma)A(\sigma)v_j d\sigma. \tag{4.7}$$

Since $v_j \to v$ and $A(\sigma)v_j = L(\sigma)u_j \to L(\sigma)M(t)^{-1}v = A(\sigma)v$ as $j \to +\infty$, we deduce that both v and $A(\sigma)v$ are in $L^p(\Omega)$. Letting $j \to +\infty$, one concludes that (4.7) holds with v in place of v_j, and the proof is complete. □

THEOREM 4.2 *Under assumptions H2, H10, H11 the solution v to the Cauchy problem (2.1) is unique.*

Proof. Let u be a solution to problem (2.1). In view of Theorem 4.1 one has

$$D_s[U(t,s)u(s)] = U(t,s)u'(s) + U(t,s)A(s)u(s) = U(t,s)f(s), \qquad (4.8)$$

if $0 \le s < t \le T_0$. Integration of both sides of (4.8) over $(0, t-\varepsilon)$, $0 < \varepsilon < t$, yields

$$U(t,t-\varepsilon)u(t-\varepsilon) - U(t,0)u_0 = \int_0^{t-\varepsilon} U(t,s)f(s)ds. \qquad (4.9)$$

By virtue of (2.25) and (2.30) one gets

$$\|U(t,t-\varepsilon) - e^{-\varepsilon A(t)}\| = \|e^{-\varepsilon A(t-\varepsilon)} + W(t,t-\varepsilon) - e^{-\varepsilon A(t)}\|$$

$$\le \|e^{-\varepsilon A(t-\varepsilon)} - e^{-\varepsilon A(t)}\| + \|W(t,t-\varepsilon)\| \le C\varepsilon^{\rho+2\beta-2}.$$

Hence

$$U(t,t-\varepsilon)u(t-\varepsilon) = U(t,t-\varepsilon)(u(t-\varepsilon) - u(t))$$

$$+(U(t,t-\varepsilon) - e^{-\varepsilon A(t)})u(t) + e^{-\varepsilon A(t)}u(t) \to u(t)$$

as $\varepsilon \to 0+$. Therefore, letting $\varepsilon \to 0+$ in (4.9), one observes that (3.5) holds, and the proof is complete. □

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5 Regularization of the solution to problem (2.1) up to the closed interval [0,T]

In this section we will show that under additional requirements on f and u_0 the solution u solves problem (2.1) in the closed interval $[0,T]$. For this purpose we need the following assumption:

H12 $f \in C^\rho([0,T_0];X)$, $v_0 \in D_0$, $A(0)v_0 - f(0) \in (D_0, L^p(\Omega))_{\beta-\gamma,p}$.

THEOREM 5.1 *Under assumptions H1, H2, H7, H12 and H10, H11, both with* $\rho = 1$*, function* v *defined by (3.25) belongs to* $C^{1+\gamma}([0,T_0];X) \cap C^\gamma([0,T_0];D_0)$ *and satisfies the Cauchy problem (3.28) in the closed interval* $[0,T_0]$*. Moreover,* v *satisfies the estimate*

$$\|v\|_{C^\gamma([0,T];D_0)} \le C(T_0)\big[\|v_0\|_{D_0} + \|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}}$$

$$+T^{\rho+\beta-1-\gamma}\|f\|_{C^\rho([0,T];X)}\big], \qquad (5.1)$$

$$\|v\|_{C^{1+\gamma}([0,T];X)} \le C(T_0)\big[\|v_0\|_{D_0} + \|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}}$$

$$+(1 + T^{\rho+\beta-1-\gamma})\|f\|_{C^\rho([0,T];X)}\big]. \qquad (5.2)$$

REMARK 5.1 Observe that u is subject to a loss of regularity in time of order $\rho - \gamma$. This fact is a strict consequence of the singularity of problem (2.1) as shown by the generation estimates (1.12) and (2.2).

To prove Theorem 5.1 we need the following Lemmata, whose proofs are postponed to the end of this Section.

LEMMA 5.1 *For any $\rho \in (1-\beta, 1)$, $\gamma \in (0, \rho+\beta-1)$ the linear operator*

$$Q_2 f(t) = \int_0^t G(t,s) f(t) ds \tag{5.3}$$

maps $C^\rho([0,T];X)$ into $C^\gamma([0,T];X)$ and satisfies the following estimate for any $T \in (0, T_0]$:

$$\|Q_2 f\|_{C^\gamma([0,T];X)} \le C(T_0) T^{\rho+\beta-1-\gamma} \|f\|_{C^\rho([0,T];X)}. \tag{5.4}$$

LEMMA 5.2 *For any $\beta \in (1/2, 1)$, $\rho \in (1-\beta, 1)$, $\gamma \in (0, \min\{2\beta-1, \rho\})$ the linear operator*

$$Q_3 f(t) = [I - e^{-tA(t)}][f(t) - f(0)] \tag{5.5}$$

maps $C^\rho([0,T];X)$ into $C^\gamma([0,T];X)$ and satisfies the following estimate for any $T \in (0, T_0]$:

$$\|Q_3 f\|_{C^\gamma([0,T];X)} \le C(T_0) T^{\rho+2\beta-1-\gamma} \|f\|_{C^\rho([0,T];X)}. \tag{5.6}$$

LEMMA 5.3 *For any $\beta \in (1/2, 1)$, $\gamma \in (0, 2\beta-1)$ and $0 \le s < t_1 < t_2 \le T$ the following estimates hold:*

$$\|A(t_2)W(t_2,s) - A(t_1)W(t_1,s)\| \le C(t_2-t_1)^\gamma (t_1-s)^{2\beta-2-\gamma}, \tag{5.7}$$

$$\|A(t_2)W(t_2,s)w_0 - A(t_1)W(t_1,s)w_0\|$$

$$\le C(t_2-t_1)^\gamma (t_2-s)^{2\beta-1-\gamma} \|w_0\|_{D_0}. \tag{5.8}$$

Proof of Theorem 5.1. We recall that v is the solution to the Cauchy problem

$$v'(t) + A(t)v(t) = f(t), \qquad v(0) = v_0 \quad 0 < t \le T_0. \tag{5.9}$$

Hence from (3.1) we get

$$\begin{aligned} v'(t) &= D_t U(t,0) w_0 + D_t \int_0^t U(t,s) f(s) ds \\ &= D_t U(t,0) w_0 + e^{-tA(t)} f(t) - \int_0^t A(s) e^{-(t-s)A(s)} [f(s) - f(t)] \\ &\quad + \int_0^t G(t,s) f(t) ds + \int_0^t D_t W(t,s) f(s) ds, \quad 0 < t \le T_0. \end{aligned}$$

Consequently,

$$A(t)v(t) = f(t) - v'(t) = f(t) - D_tU(t,0)v_0 - e^{-tA(t)}f(t)$$
$$+ \int_0^t A(s)e^{-(t-s)A(s)}[f(s) - f(t)]\,ds - \int_0^t G(t,s)f(t)ds$$
$$- \int_0^t D_tW(t,s)f(s)ds, \quad t \in (0,T].$$

Since

$$D_tU(t,0)v_0 = D_t[e^{-tA(0)} + W(t,0)]v_0$$
$$= -A(0)e^{-tA(0)}v_0 + D_tW(t,0)v_0 = -e^{-tA(0)}A(0)v_0 + D_tW(t,0)v_0,$$

we get

$$A(t)v(t) = f(t) + e^{-tA(0)}[A(0)v_0 - f(t)] - D_tW(t,0)v_0$$
$$-[e^{-tA(t)} - e^{-tA(0)}]f(t) + \int_0^t A(s)e^{-(t-s)A(s)}[f(s) - f(t)]\,ds$$
$$- \int_0^t G(t,s)f(t)ds - \int_0^t D_tW(t,s)f(s)ds =: \sum_{j=1}^{7} v_j(t).$$

We begin by estimating v_2. From the equality

$$v_2(t) = e^{-tA(0)}[A(0)v_0 - f(0)] - e^{-tA(0)}[f(t) - f(0)],$$

assumption *H12* and (3.7) one observes

$$\|e^{-tA(0)}[A(0)v_0 - f(0)]\|$$
$$\leq \|e^{-tA(0)}[A(0)v_0 - f(0)] - [A(0)v_0 - f(0)]\| + \|A(0)v_0 - f(0)\|$$
$$\leq Ct^{\gamma}\|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}} + \|A(0)v_0 - f(0)\|.$$

From this and

$$\|e^{-tA(0)}[f(t) - f(0)]\| \leq Ct^{\beta-1+\rho}|f|_{C^{\rho}([0,T];X)}$$

we immediately deduce the estimate

$$\|v_2\|_{C([0,T];X)}$$
$$\leq C(T_0)\|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}} + T^{\rho-1+\beta}|f|_{C^{\rho}([0,T];X)}. \qquad (5.10)$$

To estimate the increments of v_2 first we observe that

$$\|[e^{-t_2 A(0)} - e^{-t_1 A(0)}][A(0)v_0 - f(0)]\| = \left\| \int_{t_1}^{t_2} D_r e^{-rA(0)} [A(0)v_0 - f(0)]\, dr \right\|$$

$$= \left\| \int_{t_1}^{t_2} A(0) e^{-rA(0)} [A(0)v_0 - f(0)]\, dr \right\|$$

$$\le C \int_{t_1}^{t_2} r^{\gamma-1} \|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}} dr$$

$$\le C(t_2 - t_1)^{\gamma} \|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}}, \quad 0 < t_1 < t_2 \le T.$$

With the aid of the inequality

$$\|e^{-t_2 A(0)} - e^{-t_1 A(0)}\| = \left\| \int_{t_1}^{t_2} D_r e^{-rA(0)} dr \right\| \le C \int_{t_1}^{t_2} r^{\beta-2} dr = C \frac{t_1^{\beta-1} - t_2^{\beta-1}}{1-\beta}$$

$$= \frac{C}{1-\beta} t_1^{\beta-1} \Big[1 - \Big(\frac{t_1}{t_2} \Big)^{1-\beta} \Big] \le \frac{C}{1-\beta} t_1^{\beta-1} \Big(1 - \frac{t_1}{t_2} \Big) = \frac{C}{1-\beta} t_1^{\beta-1} \frac{t_2 - t_1}{t_2}$$

we get

$$\|e^{-t_2 A(0)} [f(t_2) - f(0)] - e^{-t_1 A(0)} [f(t_1) - f(0)]\|$$

$$= \|e^{-t_2 A(0)} [f(t_2) - f(t_1)] + [e^{-t_2 A(0)} - e^{-t_1 A(0)}][f(t_1) - f(0)]\|$$

$$\le C(T_0) |f|_{C^{\rho}([0,T];X)} \Big[t_2^{\beta-1} (t_2 - t_1)^{\rho} + t_1^{\beta-1} \frac{t_2 - t_1}{t_2} t_1^{\rho} \Big]$$

$$\le C(T_0) |f|_{C^{\rho}([0,T];X)} \Big[\Big(\frac{t_2 - t_1}{t_2} \Big)^{1-\beta} (t_2 - t_1)^{\rho-1+\beta} + t_1^{\rho+\beta-1} \Big(\frac{t_2 - t_1}{t_2} \Big)^{\rho-1+\beta} \Big]$$

$$= C(T_0) |f|_{C^{\rho}([0,T];X)} \Big[\Big(\frac{t_2 - t_1}{t_2} \Big)^{1-\beta} + \Big(\frac{t_1}{t_2} \Big)^{\rho+\beta-1} \Big] (t_2 - t_1)^{\rho-1+\beta}$$

$$\le C(T_0) (t_2 - t_1)^{\rho-1+\beta} |f|_{C^{\rho}([0,T];X)}.$$

Since $\rho > 1 - \beta + \gamma$, we have shown that

$$|v_2|_{C^{\gamma}([0,T];X)} \le C(T_0) [\|A(0)v_0 - f(0)\|_{(D_0,X)_{\beta-\gamma,p}} + T^{\rho-1+\beta-\gamma} |f|_{C^{\rho}([0,T];X)}].$$

We now estimate v_3. From the formula (cf. (2.42))

$$v_3(t) = -D_t W(t,0) v_0 = A(t) W(t,0) v_0 + \Phi_1(t,0) v_0$$

it follows

$$\|v_3(t_2) - v_3(t_1)\|$$

$$= \|A(t_2) W(t_2,0) v_0 - A(t_1) W(t_1,0) v_0 - \Phi_1(t_2,0) v_0 + \Phi_1(t_1,0) v_0\|.$$

With the aid of the following inequalities, where $\gamma \in (0, 2\beta - 1)$,

$$\| [A(t_2)W(t_2,s) - A(t_1)W(t_1,s)] v_0 \| \le C(t_2-t_1)^\gamma (t_2-s)^{2\beta-1-\gamma}\|v_0\|_{D_0},$$

$$\|[\Phi_1(t_2,s) - \Phi_1(t_1,s)]v_0\| \le C(t_2-t_1)^\gamma (t_2-s)^{\beta-\gamma}\|v_0\|_{D_0}$$

(cf. Lemma 5.1 and Lemma 2.2, with $\rho = 1$ and $\nu = \gamma$), we easily obtain the estimate

$$|v_3|_{C^\gamma([0,T];X)} \le C\|v_0\|_{D_0}(T^{2\beta-1-\gamma} + T^{\beta-\gamma})$$

proving the continuity of v_3 in $[0,T]$. To deduce the estimate for v_3 in $C([0,T],X)$ it is enough to use the previous representation for v_3 and estimate (2.35). We get

$$\|v_3\|_{C([0,T];X)} \le C\|v_0\|_{D_0} T^{\rho-2\beta-2}.$$

We now estimate v_4. In view of (2.17) with $\tau = t$, $s = 0$, $\rho = 1$ one has

$$\|e^{-tA(t)} - e^{-tA(0)}\| \le Ct^{2\beta-1}.$$

Hence

$$\|v_4(t)\| \le Ct^{2\beta-1}\|f(t)\|,$$

which implies

$$\|v_4\|_{C([0,T];X)} \le T^{2\beta-1}\|f\|_{C([0,T];X)}.$$

Consider next the relations

$$\begin{aligned}
&(e^{-t_2A(t_2)} - e^{-t_2A(0)}) - (e^{-t_1A(t_1)} - e^{-t_1A(0)})\\
&= \frac{1}{2\pi i}\int_\Gamma e^{t_2\lambda}\{(\lambda + A(t_2))^{-1} - (\lambda + A(0))^{-1}\}d\lambda\\
&\quad - \frac{1}{2\pi i}\int_\Gamma e^{t_1\lambda}\{(\lambda + A(t_1))^{-1} - (\lambda + A(0))^{-1}\}d\lambda\\
&= \frac{1}{2\pi i}\int_\Gamma e^{t_2\lambda}\{(\lambda + A(t_2))^{-1} - (\lambda + A(t_1))^{-1}\}d\lambda\\
&\quad + \frac{1}{2\pi i}\int_\Gamma (e^{t_2\lambda} - e^{t_1\lambda})\{(\lambda + A(t_1))^{-1} - (\lambda + A(0))^{-1}\}d\lambda.
\end{aligned}$$

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int_\Gamma e^{t_2\lambda}\{(\lambda + A(t_2))^{-1} - (\lambda + A(t_1))^{-1}\}d\lambda\right\|\\
&\le C\int_\Gamma e^{t_2\mathrm{Re}\lambda}|\lambda|^{-\beta}(t_2-t_1)|\lambda|^{1-\beta}|d\lambda| = C(t_2-t_1)\int_\Gamma e^{t_2\mathrm{Re}\lambda}|\lambda|^{1-2\beta}|d\lambda|\\
&\le C(t_2-t_1){t_2}^{2\beta-2} = C\frac{t_2-t_1}{t_2}{t_2}^{2\beta-1} \le C\left(\frac{t_2-t_1}{t_2}\right)^\gamma {t_2}^{2\beta-1}\\
&= Ct_2^{2\beta-1-\gamma}(t_2-t_1)^\gamma.
\end{aligned}$$

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int_\Gamma (e^{t_2\lambda}-e^{t_1\lambda})\{(\lambda+A(t_1))^{-1}-(\lambda+A(0))^{-1}\}d\lambda\right\| \\
&=\left\|\frac{1}{2\pi i}\int_\Gamma\int_{t_1}^{t_2}\lambda e^{r\lambda}dr\{(\lambda+A(t_1))^{-1}-(\lambda+A(0))^{-1}\}d\lambda\right\| \\
&=\left\|\frac{1}{2\pi i}\int_{t_1}^{t_2}\int_\Gamma \lambda e^{r\lambda}\{(\lambda+A(t_1))^{-1}-(\lambda+A(0))^{-1}\}d\lambda dr\right\| \\
&\le C\int_{t_1}^{t_2}\int_\Gamma |\lambda| e^{r\mathrm{Re}\lambda}|\lambda|^{-\beta}t_1|\lambda|^{1-\beta}|d\lambda|dr = Ct_1\int_{t_1}^{t_2}\int_\Gamma e^{r\mathrm{Re}\lambda}|\lambda|^{2-2\beta}|d\lambda|dr \\
&\le Ct_1\int_{t_1}^{t_2} r^{2\beta-3}dr \le Ct_1{t_2}^{2\beta-1}\int_{t_1}^{t_2} r^{-2}dr = Ct_1{t_2}^{2\beta-1}\frac{t_2-t_1}{t_2t_1} \\
&= C{t_2}^{2\beta-1}\frac{t_2-t_1}{t_2} \le C{t_2}^{2\beta-1}\left(\frac{t_2-t_1}{t_2}\right)^\gamma = C{t_2}^{2\beta-1-\gamma}(t_2-t_1)^\gamma.
\end{aligned}$$

Hence

$$\|[e^{-t_2A(t_2)}-e^{-t_2A(0)}]-[e^{-t_1A(t_1)}-e^{-t_1A(0)}]\| \le Ct_2^{2\beta-1-\gamma}(t_2-t_1)^\gamma.$$

With the aid of this inequality one concludes

$$\begin{aligned}
&\|v_4(t_2)-v_4(t_1)\| \\
&\le \|\{(e^{-t_2A(t_2)}-e^{-t_2A(0)})-(e^{-t_1A(t_1)}-e^{-t_1A(0)})\}f(t_2)\| \\
&\quad + \|(e^{-t_1A(t_1)}-e^{-t_1A(0)})(f(t_2)-f(t_1))\| \\
&\le C{t_2}^{2\beta-1-\gamma}(t_2-t_1)^\gamma\|f(t_2)\| + Ct_1^{2\beta-1}(t_2-t_1)^\rho|f|_{C^\rho([0,T];X)}.
\end{aligned}$$

We have thus shown the estimate

$$|v_4|_{C^\gamma([0,T];X)} \le C(T_0)(T^{2\beta-1-\gamma}+T^{2\beta-1-\gamma+\rho})\|f\|_{C^\rho([0,T];X)}.$$

We now estimate v_5. Observe that

$$v_5(t)=\int_0^t A(s)e^{-(t-s)A(s)}[f(s)-f(t)]ds = Q_2f(t).$$

Then from Lemma 5.2 we deduce

$$\|v_5\|_{C^\gamma([0,T];X)} \le C(t_2-t_1)^\gamma T^{\rho+\beta-1-\gamma}|f|_{C^\rho([0,T];X)}.$$

We now estimate v_6. From the identity

$$\begin{aligned}
v_6(t) &= -\int_0^t G(t,s)[f(t)-f(0)]ds - \int_0^t G(t,s)f(0)ds \\
&= Q_3(f-f(0))(t) - \int_0^t G(t,s)f(0)ds
\end{aligned}$$

it follows

$$\|v_6(t_2) - v_6(t_1)\| \leq \|Q_3(f - f(0))(t_2) - Q_3(f - f(0))(t_1)\| \\ + \left\| \int_0^{t_2} G(t_2, s)f(0)ds - \int_0^{t_1} G(t_1, s)f(0)ds \right\|.$$

Then since $0 < \gamma < \min\{2\beta - 1, \rho\}$, Lemma 5.3 yields

$$\|Q_3(f - f(0))(t_2) - Q_3(f - f(0))(t_1)\| \\ \leq C(T_0)(t_2 - t_1)^\gamma T^{\rho+2\beta-1-\gamma}|f|_{C^\rho([0,T];X)}.$$

On the other hand, from (2.16), with $\rho = 1$, and (5.21), with $\varepsilon = \gamma$, since $0 < \gamma < 2\beta - 1$, we deduce

$$\|v_6(t)\| \leq C\|f\|_{C([0,T];X)} \int_0^t (t - s)^{2\beta-2} ds = C\|f\|_{C([0,T];X)} \frac{t^{2\beta-1}}{2\beta - 1},$$

$$\left\| \int_0^{t_2} G(t_2, s)f(0)ds - \int_0^{t_1} G(t_1, s)f(0)ds \right\|$$

$$= \left\| \int_{t_1}^{t_2} G(t_2, s)f(0)ds + \int_0^{t_1} (G(t_2, s) - G(t_1, s))f(0)ds \right\|$$

$$\leq C\|f(0)\| \Big\{ \int_{t_1}^{t_2} (t_2 - s)^{2\beta-2} ds + (t_2 - t_1)^\gamma \int_0^{t_1} (t_1 - s)^{2\beta-\gamma-2} ds \Big\}$$

$$= C\|f(0)\| \Big\{ \frac{(t_2 - t_1)^{2\beta-1}}{2\beta - 1} + (t_2 - t_1)^\gamma \frac{t_1^{2\beta-\gamma-1}}{2\beta - \gamma - 1} \Big\}$$

$$\leq C\|f(0)\|(t_2 - t_1)^\gamma T^{2\beta-\gamma-1}.$$

Therefore

$$\|v_6\|_{C([0,T];X)} \leq C(T_0)\|f\|_{C([0,T];X)} T^{2\beta-1},$$

$$|v_6|_{C^\gamma([0,T];X)} \leq C(T_0)(T^{\rho+2\beta-1-\gamma} + T^{2\beta-\gamma-1})[\|f(0)\| + |f|_{C^\rho([0,T];X)}].$$

Finally, we estimate v_7. Since

$$D_t W(t, s) + A(t)W(t, s) = -\Phi_1(t, s)$$

(cf. (2.42)), we get

$$v_7(t) = \int_0^t A(t)W(t, s)f(s)ds + \int_0^t \Phi_1(t, s)f(s)ds.$$

From the inequalities (2.40) with $\rho = 1$ and (5.7) it follows

$$\begin{aligned}
&\left\| \int_0^{t_2} A(t_2)W(t_2,s)f(s)ds - \int_0^{t_1} A(t_1)W(t_1,s)f(s)ds \right\| \\
&= \left\| \int_{t_1}^{t_2} A(t_2)W(t_2,s)f(s)ds + \int_0^{t_1} [A(t_2)W(t_2,s) - A(t_1)W(t_1,s)]f(s)ds \right\| \\
&\le C\|f\|_{C([0,T];X)} \Big\{ \int_{t_1}^{t_2} (t_2-s)^{2\beta-2} ds + (t_2-t_1)^\gamma \int_0^{t_1} (t_1-s)^{2\beta-2-\gamma} ds \Big\} \\
&= C\|f\|_{C([0,T];X)} \Big\{ \frac{(t_2-t_1)^{2\beta-1}}{2\beta-1} + (t_2-t_1)^\gamma \frac{t_1^{2\beta-1-\gamma}}{2\beta-1-\gamma} \Big\},
\end{aligned}$$

and from (2.11) and (2.46), with $\rho = 1$ and $\nu = \gamma$, we get

$$\begin{aligned}
&\left\| \int_0^{t_2} \Phi_1(t_2,s)f(s)ds - \int_0^{t_1} \Phi_1(t_1,s)f(s)ds \right\| \\
&= \left\| \int_{t_1}^{t_2} \Phi_1(t_2,s)f(s)ds + \int_0^{t_1} [\Phi_1(t_2,s) - \Phi_1(t_1,s)]f(s)ds \right\| \\
&\le C\|f\|_{C([0,T];X)} \Big\{ \int_{t_1}^{t_2} (t_2-s)^{\beta-1} ds + \int_0^{t_1} (t_2-t_1)^\gamma (t_1-s)^{\beta-1-\gamma} ds \Big\} \\
&= C\|f\|_{C([0,T];X)} \Big\{ \frac{(t_2-t_1)^{\beta}}{\beta} + (t_2-t_1)^\gamma \frac{t_1^{\beta-\gamma}}{\beta-\gamma} \Big\}.
\end{aligned}$$

Therefore

$$|v_7|_{C^\gamma([0,T];X)} \le C(T^{2\beta-1-\gamma} + T^{\beta-\gamma})\|f\|_{C([0,T];X)}.$$

Likewise, from (2.11) and (2.40), both with $\rho = 1$, we get

$$\|v_7\|_{C([0,T];X)} \le C(T^{2\beta-1} + T^{\beta})\|f\|_{C([0,T];X)}.$$

Summing up, we have proved that $A(\cdot)v \in C^\gamma([0,T_0];X)$ and can be estimated by the right-hand side in (5.1). Likewise, owing to formula (5.9) we can show that $v' \in C^\gamma([0,T_0];X)$ and can be estimated by the right-hand side in (5.2).

We conclude the proof by observing that the estimate of v in $C([0,T_0];X)$ was proved in Theorem 3.2 (cf. estimate (3.27)). □

Proof of Lemma 5.1. From the definition (5.24) of operator Q_2 and estimate (2.8) we easily deduce the inequality

$$\begin{aligned}
\|Q_2 f(t)\| &\le C\|f\|_{C^\rho([0,T];X)} \int_0^t (t-s)^{\rho+\beta-2}\, ds \\
&\le CT^{\rho+\beta-1}\|f\|_{C^\rho([0,T];X)}. \qquad (5.11)
\end{aligned}$$

To estimate the increments of Q_2f we need the following identities

$$Q_2f(t_2) - Q_2f(t_1) = \int_{t_1}^{t_2} A(s)e^{-(t_2-s)A(s)}[f(s) - f(t_2)]\, ds$$
$$+ \int_0^{t_1} [A(s)e^{-(t_2-s)A(s)} - A(s)e^{-(t_1-s)A(s)}][f(s) - f(t_1)]\, ds$$
$$+ \int_0^{t_1} A(s)e^{-(t_2-s)A(s)}[f(t_1) - f(t_2)]\, ds =: \sum_{j=1}^{3} Q_{2,j}(t_1,t_2). \quad (5.12)$$

Observe now that

$$\frac{t_2 - t_1}{t_2 - s}(t_1 - s)^{\rho+\beta-2} \le (t_2 - t_1)^{\gamma}(t_1 - s)^{\rho+\beta-2-\gamma}, \quad 0 \le s < t_1 < t_2 \le T. \quad (5.13)$$

The assertion now follows from (2.17), with $(\tau, t) = (t_1, t_2)$, (5.13) and the inequality $\gamma \in (0, \rho + \beta - 1)$. Indeed, we get

$$\|Q_{2,1}(t_1,t_2)\| \le C\|f\|_{C^{\rho}([0,T];X)} \int_{t_1}^{t_2} (t_2 - s)^{\gamma}(t_2 - s)^{\rho+\beta-2-\gamma}\, ds$$
$$\le C(t_2 - t_1)^{\gamma}\|f\|_{C^{\rho}([0,T];X)} \int_0^{t_2} (t_2 - s)^{\rho+\beta-2-\gamma}\, ds$$
$$\le C(t_2 - t_1)^{\gamma}T^{\rho+\beta-1-\gamma}\|f\|_{C^{\rho}([0,T];X)}, \quad (5.14)$$

$$\|Q_{2,2}(t_1,t_2)\| \le C\|f\|_{C^{\rho}([0,T];X)}(t_2 - t_1)^{\gamma} \int_0^{t_1} (t_1 - s)^{\rho+\beta-2-\gamma}\, ds$$
$$\le C(t_2 - t_1)^{\gamma}T^{\rho+\beta-1-\gamma}\|f\|_{C^{\rho}([0,T];X)}, \quad (5.15)$$

$$\|Q_{2,3}(t_1,t_2)\| \le (t_2 - t_1)^{\gamma}\|f\|_{C^{\rho}([0,T];X)} \int_0^{t_1} (t_2 - s)^{\beta-2}(t_2 - t_1)^{\rho-\gamma}\, ds$$
$$\le (t_2 - t_1)^{\gamma}\|f\|_{C^{\rho}([0,T];X)} \int_0^{t_1} (t_2 - s)^{\rho+\beta-2-\gamma}\, ds$$
$$\le (t_2 - t_1)^{\gamma}T^{\rho+\beta-1-\gamma}\|f\|_{C^{\rho}([0,T];X)}. \quad (5.16)$$

Proof of Lemma 5.2. Let $0 \le s < t \le T$. From (2.14) we easily deduce the formula

$$D_tG(t,s) = D_t[A(t)e^{-(t-s)A(t)} - A(s)e^{-(t-s)A(s)}]$$
$$= \frac{1}{2\pi i}\int_{\Gamma} \lambda^2 e^{\lambda(t-s)} \left((\lambda + A(t))^{-1} - (\lambda + A(s))^{-1}\right) d\lambda$$
$$+ \frac{1}{2\pi i}\int_{\Gamma} \lambda e^{\lambda(t-s)} \left((\lambda + A(t))^{-1}A'(t)A(t)^{-1}A(t)(\lambda + A(t))^{-1}\right) d\lambda. \quad (5.17)$$

Consider now the identity

$$A'(t)A(t)^{-1} = L'(t)L(t)^{-1} - L(t)[M_1(t)^{-1}M_1'(t)]L(t)^{-1} \tag{5.18}$$

and note that, according to assumptions *H1* and *H4*,

$$\|A'(t)A(t)^{-1}\| \le C(T_0), \qquad t \in [0, T_0]. \tag{5.19}$$

Then from (5.17), (5.19) and (2.15), with $\rho = 1$, we deduce the estimates

$$\begin{aligned}
&\|D_t G(t,s)\| = \|D_t[A(t)e^{-(t-s)A(t)} - A(s)e^{-(t-s)A(s)}]\| \\
&\le C(T_0)(t-s)\int_\Gamma |\lambda|^{3-2\beta} e^{(t-s)\mathrm{Re}\,\lambda}\,|d\lambda| + C(T_0)\int_\Gamma |\lambda|^{2-2\beta} e^{(t-s)\mathrm{Re}\,\lambda}\,|d\lambda| \\
&\le C(T_0)(t-s)^{2\beta-3}.
\end{aligned} \tag{5.20}$$

Whence, for any $\varepsilon \in \mathbf{R}_+$, we easily deduce the estimates

$$\begin{aligned}
&\|G(t_2,s) - G(t_1,s)\| \le \int_{t_1}^{t_2} \|D_t G(t,s)\|\,dt \le C(T_0)\int_{t_1}^{t_2} (t-s)^{2\beta-3}\,dt \\
&\le C(T_0)(t_1-s)^{2\beta-\varepsilon-2}\int_{t_1}^{t_2} (t-s)^{\varepsilon-1}\,dt \\
&= C(T_0)(t_1-s)^{2\beta-\varepsilon-2}(t_2-t_1)^{\varepsilon}.
\end{aligned} \tag{5.21}$$

Moreover, since $\beta \in (1/2, 1)$ and $\gamma \in (0, \min\{2\beta-1, \rho\})$ (cf. *H7*), we get

$$\begin{aligned}
&\left\|\int_{t_1}^{t_2} G(t_2,s)f(t_2)ds\right\| \le C(T_0)\int_{t_1}^{t_2} (t_2-s)^{2\beta-2}\|f(t_2)\|\,ds \\
&= C(T_0)\frac{(t_2-t_1)^{2\beta-1}}{2\beta-1}\|f(t_2)-f(0)\| \le C(T_0)\frac{(t_2-t_1)^{2\beta-1}}{2\beta-1}t_2^{\rho}|f|_{C^{\rho}([0,T];X)} \\
&= C(T_0)\frac{(t_2-t_1)^{\gamma}(t_2-t_1)^{2\beta-1-\gamma}}{2\beta-1}t_2^{\rho}|f|_{C^{\rho}([0,T];X)} \\
&\le C(T_0)(t_2-t_1)^{\gamma}T^{\rho+2\beta-1-\gamma}|f|_{C^{\rho}([0,T];X)},
\end{aligned}$$

$$\left\| \int_0^{t_1} [G(t_2, s) - G(t_1, s)] f(t_1) ds \right\|$$

$$\leq C(T_0) \int_0^{t_1} (t_2 - t_1)^\gamma (t_1 - s)^{2\beta - 2 - \gamma} ds \, \|f(t_1) - f(0)\|$$

$$\leq C(T_0)(t_2 - t_1)^\gamma \frac{t_1^{2\beta - 1 - \gamma}}{2\beta - 1 - \gamma} t_1^\rho |f|_{C^\rho([0,T];X)}$$

$$\leq C(T_0)(t_2 - t_1)^\gamma T^{\rho + 2\beta - 1 - \gamma} |f|_{C^\rho([0,T];X)}, \tag{5.22}$$

$$\left\| \int_0^{t_1} G(t_2, s)[f(t_2) - f(t_1)] ds \right\| \leq C(T_0) \int_0^{t_1} (t_2 - s)^{2\beta - 2} ds \, (t_2 - t_1)^\rho |f|_{C^\rho([0,T];X)}$$

$$\leq C(T_0) \frac{t_2^{2\beta - 1}}{2\beta - 1} (t_2 - t_1)^\gamma (t_2 - t_1)^{\rho - \gamma} |f|_{C^\rho([0,T];X)}$$

$$\leq C(T_0)(t_2 - t_1)^\gamma T^{\rho + 2\beta - 1 - \gamma} |f|_{C^\rho([0,T];X)}. \tag{5.23}$$

Finally, the assertion of the lemma easily follows from (5.22)-(5.23) and the inequalities

$$\|Q_3 f(t)\| \leq \int_0^t \|G(t, s)\| \, ds \, \|f(t)\| \leq C(T_0) \|f(t) - f(0)\| \int_0^t (t - s)^{2\beta - 2} \, ds$$

$$\leq C(T_0) \frac{t^{2\beta - 1}}{2\beta - 1} t^\rho |f|_{C^\rho([0,T];X)} \leq C(T_0) T^{\rho + 2\beta - 1} |f|_{C^\rho([0,T];X)},$$

$$\|Q_3 f(t_2) - Q_3 f(t_1)\| \leq \int_{t_1}^{t_2} \|G(t_2, s)\| \, ds \, \|f(t_2)\|$$

$$+ \int_0^{t_1} \|G(t_2, s) - G(t_1, s)\| \, ds \, \|f(t_1)\| + \int_0^{t_1} \|G(t_2, s)\| \, ds \, \|f(t_2) - f(t_1)\|$$

$$\leq C(T_0)(t_2 - t_1)^\gamma T^{\rho + 2\beta - 1 - \gamma} |f|_{C^\rho([0,T];X)}, \quad t \in [0, T]. \ \square$$

Proof of Lemma 5.3. Recalling that $W(t, s)$ is defined by (2.28), we can show analogously to (2.38) that function $A(t)W(t, s)$ is represented by the following formula, where $0 < s < t < T$:

$$A(t)W(t, s) = \int_s^t \Phi_1(t, \tau)\Phi(\tau, s) d\tau + \int_s^t A(\tau) e^{-(t - \tau)A(\tau)} [\Phi(\tau, s) - \Phi(t, s)] \, d\tau$$

$$- \int_s^t G(t, \tau) \, d\tau \, \Phi(t, s) + [I - e^{-(t - s)A(t)}] \Phi(t, s) =: \sum_{j=1}^4 W_j(t, s). \tag{5.24}$$

Taking Lemma 3.1 and the definitions of operators W_j, $j = 1, \dots, 4$, into account, we easily derive the identities

$$W_1(t_2,s) - W_1(t_1,s)$$

$$= \int_{t_1}^{t_2} \Phi_1(t_2,\tau)\Phi(\tau,s)d\tau + \int_s^{t_1} [\Phi_1(t_2,\tau) - \Phi_1(t_1,\tau)]\Phi(\tau,s)d\tau,$$

$$W_2(t_2,s) - W_2(t_1,s) = \int_{t_1}^{t_2} A(\tau)e^{-(t_2-\tau)A(\tau)}[\Phi(\tau,s) - \Phi(t_2,s)]\,d\tau$$

$$+ \int_s^{t_1} A(\tau)e^{-(t_2-\tau)A(\tau)}[\Phi(t_1,s) - \Phi(t_2,s)]\,d\tau$$

$$+ \int_s^{t_1} \big[A(\tau)e^{-(t_1-\tau)A(\tau)} - A(\tau)e^{-(t_2-\tau)A(\tau)}\big][\Phi(t_1,s) - \Phi(\tau,s)]\,d\tau, \quad (5.25)$$

$$W_3(t_2,s) - W_3(t_1,s) = \int_{t_1}^{t_2} G(t_2,\tau)\,d\tau\,\Phi(t_2,s)$$

$$+ \int_s^{t_1} [G(t_2,\tau) - G(t_1,\tau)]\,d\tau\,\Phi(t_2,s)$$

$$+ \int_s^{t_1} G(t_1,\tau)\,d\tau\,\big[\Phi(t_2,s) - \Phi(t_1,s)\big], \quad (5.26)$$

$$W_4(t_2,s) - W_4(t_1,s) = [e^{-(t_1-s)A(t_1)} - e^{-(t_2-s)A(t_2)}]\Phi(t_1,s)$$

$$+[I - e^{-(t_2-s)A(t_2)}]\big[\Phi(t_2,s) - \Phi(t_1,s)\big]. \quad (5.27)$$

From (5.25)-(5.27) and (2.11), (2.12), (2.46), (2.21), (2.18), (2.16), (5.21) and the inequalities

$$\|e^{-(t_2-s)A(t_2)} - e^{-(t_1-s)A(t_1)}\| \le \int_{t_1}^{t_2} \|D_t e^{-(t-s)A(t)}\|\,dt$$

$$\le C(T_0)\int_{t_1}^{t_2} (t-s)^{\beta-2}\,dt \le C(T_0)(t_1-s)^{\beta-1-\gamma}\int_{t_1}^{t_2} (t-s)^{\gamma-1}\,dt$$

$$\le C(T_0)(t_1-s)^{\beta-1-\gamma}(t_2-t_1)^{\gamma}, \quad (5.28)$$

we easily derive the following estimates, where $\beta \in (1/2, 1)$, $\gamma \in (0, 2\beta - 1)$, $\nu \in (1-\beta+\gamma, \beta)$:

$$\|W_1(t_2,s) - W_1(t_1,s)\| \le C(T_0) \int_{t_1}^{t_2} (t_2-\tau)^{\gamma}(t_2-\tau)^{\beta-1-\gamma}(\tau-s)^{\beta-1}\,d\tau$$

$$+C(T_0)(t_2-t_1)^{\gamma} \int_s^{t_1} (t_1-\tau)^{\beta-1-\gamma}(\tau-s)^{\beta-1}d\tau$$

$$\le C(T_0)(t_2-t_1)^{\gamma} \sum_{j=1}^{2} \int_s^{t_j} (t_j-\tau)^{\beta-1-\gamma}(\tau-s)^{\beta-1}\,d\tau$$

$$= C(T_0)(t_2-t_1)^{\gamma} \sum_{j=1}^{2} (t_j-s)^{2\beta-1-\gamma} \le C(T_0)(t_2-t_1)^{\gamma}(t_2-s)^{2\beta-1-\gamma}, \quad (5.29)$$

$$\|W_2(t_2,s) - W_2(t_1,s)\| \le C(T_0)\Big\{ \int_{t_1}^{t_2} (t_2-\tau)^{\beta-2+\nu}(\tau-s)^{\beta-1-\nu}\,d\tau$$

$$+(t_2-t_1)^{\gamma}(t_1-s)^{\beta-1-\nu} \int_s^{t_1} (t_2-\tau)^{\beta-2}(t_2-t_1)^{\nu-\gamma}\,d\tau$$

$$+(t_2-t_1)^{\gamma} \int_s^{t_1} (t_1-\tau)^{\beta-2-\gamma}(t_1-\tau)^{\nu}(\tau-s)^{\beta-1-\nu}\,d\tau\Big\}$$

$$\le C(T_0)(t_2-t_1)^{\gamma}\Big\{ \int_{t_1}^{t_2} (t_2-\tau)^{\beta-2+\nu-\gamma}(\tau-s)^{\beta-1-\nu}\,d\tau$$

$$+ (t_1-s)^{\beta-1-\nu} \int_s^{t_1} (t_2-\tau)^{\beta-2-\gamma+\nu}\,d\tau$$

$$+ \int_s^{t_1} (t_1-\tau)^{\beta-2-\gamma+\nu}(\tau-s)^{\beta-1-\nu}\,d\tau\Big\} \le C(T_0)(t_2-t_1)^{\gamma}\Big\{(t_2-s)^{2\beta-2-\gamma}$$

$$+ (t_1-s)^{\beta-1-\nu} \int_s^{t_1} (t_1-\tau)^{\beta-2-\gamma+\nu}\,d\tau + (t_1-s)^{2\beta-2-\gamma}\Big\}$$

$$\le C(T_0)(t_2-t_1)^{\gamma}(t_1-s)^{2\beta-2-\gamma}, \quad (5.30)$$

$$\|W_3(t_2,s) - W_3(t_1,s)\| \le C(T_0)\Big\{(t_2-s)^{\beta-1} \int_{t_1}^{t_2} (t_2-\tau)^{\gamma}(t_2-\tau)^{2\beta-2-\gamma}\,d\tau$$

$$+ (t_2-t_1)^{\gamma}\Big[(t_2-s)^{\beta-1} \int_s^{t_1} (t_1-\tau)^{2\beta-\gamma-2}\,d\tau$$

$$+ (t_1-s)^{\beta-1-\gamma} \int_s^{t_1} (t_1-\tau)^{2\beta-2}\,d\tau\Big]\Big\}$$

$$\le C(T_0)(t_2-t_1)^{\gamma}\Big\{(t_2-s)^{\beta-1} \int_s^{t_2} (t_2-\tau)^{2\beta-2-\gamma}\,d\tau$$

$$+ (t_1-s)^{2\beta-\gamma-1}(t_2-s)^{\beta-1} + (t_1-s)^{2\beta-1}(t_1-s)^{\beta-1-\gamma}\Big\}$$

$$\leq C(T_0)(t_2-t_1)^\gamma \sum_{j=1}^{2} (t_j - s)^{3\beta-2-\gamma} \leq C(T_0)(t_2-t_1)^\gamma (t_1-s)^{3\beta-2-\gamma}, \quad (5.31)$$

$$\|W_4(t_2,s) - W_4(t_1,s)\| \leq C(t_2-t_1)^\gamma (t_1-s)^{2\beta-2-\gamma}. \quad (5.32)$$

From identities (5.24), (5.25)–(5.27) and estimates (5.29)–(5.32) we easily derive (5.7). The inequality (5.8) can be shown analogously using (2.22) and (2.23) instead of (2.12) and (2.21). □

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6 An equivalent identification problem and proof of Theorem 1.1

Throughout this section we will use the estimates from Section 2 with $\rho = 1$. To find an identification problem equivalent to (1.6)–(1.9) first we consider the equation

$$g''(t) = \Psi[D_t(m(t)w(t))] + \Psi\Big[D_t^2 m(t)\Big(u_0 + \int_0^t w(s)\,ds\Big)\Big]$$
$$+\Psi[D_t m(t) w(t)], \quad t \in [0,T]. \quad (6.1)$$

Assume that

$$\chi^{-1} := \Psi[B(0)u_0] \neq 0, \quad (6.2)$$

Applying functional Ψ to both sides in (1.6) we derive the following equation for k:

$$k(t) = \chi\{\Psi[D_t f(t)] - g''(t)\} + \chi\Psi\Big[D_t^2 m(t)\Big(u_0 + \int_0^t w(s)\,ds\Big)\Big]$$
$$+\chi\Psi[(D_t m(t) - L(t))w(t)] - \chi\Psi\Big[L'(t)\Big(u_0 + \int_0^t w(s)\,ds\Big)(x)\Big]$$
$$-\chi\int_0^t k(t-s)\Psi[B(s)w(s)]\,ds$$
$$-\chi\int_0^t k(t-s)\Psi\Big[B'(s)\Big(u_0 + \int_0^s w(r)\,dr\Big)ds\Big]. \quad (6.3)$$

Introduce the new unknown

$$z(t) = L(t)w(t) \quad \Longleftrightarrow \quad w(t) = L(t)^{-1}z(t), \qquad 0 \leq t \leq T, \quad (6.4)$$

Then, according to the Theorem 3.2, we easily deduce that the initial and boundary value problem (1.6)–(1.8) is equivalent to the following operator equation:

$$\begin{aligned} z(t) &= A(t)M(t)w(t) \\ &= z_0(t) - Q[L_1 z + L_2 k + B_2(z,k)](t) - L_3 k(t), \\ &= z_0(t) + N_1(z,k)(t), \quad t \in (0,T). \end{aligned} \tag{6.5}$$

Here we have set

$$z_0(t) = A(t)\Big\{U(t,0)w_0 + \int_0^t U(t,s)[f'(s) - L'(s)u_0]\,ds\Big\}, \tag{6.6}$$

$$Qf(t) = A(t)\int_0^t U(t,s)f(s)\,ds \tag{6.7}$$

$$L_1 z(t) = \int_0^t L'(t)L(s)^{-1}z(s)\,ds, \tag{6.8}$$

$$L_2 k(t) = \int_0^t k(t-s)B'(s)u_0\,ds, \tag{6.9}$$

$$L_3(k)(t) = \int_0^t k(s)A(t)U(t,s)B(0)u_0\,ds, \tag{6.10}$$

$$B_2(z,k)(t) = \int_0^t k(t-s)\Big(B(s)L(s)^{-1}z(s) + B'(s)\int_0^s L(r)^{-1}z(r)\,dr\Big)ds. \tag{6.11}$$

After some simple computations, from system (6.3), (6.5) we derive the following equation for k:

$$k(t) = k_0(t) + N_2(z,k)(t), \qquad t \in [0,T], \tag{6.12}$$

where we have set

$$\begin{aligned} k_0(t) &= \chi\{\Psi[D_t f(t)] - g''(t)\} + \chi\Psi[D_t^2 m(t)u_0] \\ &\quad + \chi\Psi[(D_t m(t)L(t)^{-1} - 1)z_0(t)] - \chi\Psi[L'(t)u_0], \end{aligned} \tag{6.13}$$

$$\begin{aligned} N_2(z,k)(t) &= \chi\Psi[L_4 z(t)] + \chi\Psi[(D_t m(t)L(t)^{-1} - 1)N_1(z,k)(t)] \\ &\quad - \chi\Psi[L_1 z(t)] - \chi\Psi[L_2 k(t)] - \chi\Psi[B_2(z,k)(t)], \quad t \in [0,T], \end{aligned} \tag{6.14}$$

and

$$L_4 z(t) = D_t^2 m(t)\int_0^t L(s)^{-1}z(s)\,ds. \tag{6.15}$$

Observe that system (6.5), (6.12) is equivalent to system (6.3), (6.5) via (6.4).

Our main task consists in showing that the fixed point-system (6.5), (6.12) is solvable in $C^\gamma([0,T];X) \times C^\gamma([0,T];\mathbf{R})$ for some γ satisfying *H7*.

To estimate N_1 it is convenient to estimate in $C^\gamma([0,\tau];X)$, $\tau \in (0,T]$ first the linear operator Q. From Theorem 5.1, with $u_0 = f(0) = 0$, we easily deduce the estimate

$$\|Qf\|_{C^\gamma([0,T];X)} \leq C(T_0)T^{\rho+\beta-1-\gamma}\|f\|_{C^\rho([0,T];X)}. \tag{6.16}$$

REMARK 6.1 We stress that applying operator Q causes a loss in regularity of order $\rho - \gamma$, at most. Therefore our basic task consists in restoring regularity.

In order to estimate the nonlinear operator N_1 (cf. (6.5)) we need the following lemmata 6.1-6.5. The proofs of the first three will be postponed to the end of this Section.

LEMMA 6.1 *The linear operator L_1 and L_4 defined by (6.8) and (6.15) map $C([0,T];X)$ into $C^\rho([0,T];X)$ and $C^\gamma([0,T];X)$, respectively, and satisfy the estimates*

$$\|L_1 z\|_{C^\rho([0,T];X)} \leq C(T_0)T^{1-\rho}\|z\|_{C([0,T];X)}, \tag{6.17}$$

$$\|L_4 z\|_{C^\gamma([0,T];X)} \leq C(T_0)T^{1-\gamma}\|z\|_{C([0,T];X)}. \tag{6.18}$$

LEMMA 6.2 *Operators B_2 and L_2 defined by (6.11) and (6.9) map $C^{\rho-\gamma}([0,T];X) \times C^\gamma([0,T];\mathbf{R})$ and $C^\gamma([0,T];X)$, respectively, into $C^\rho([0,T];X)$ and satisfy the following estimates for any $T \in (0,T_0]$:*

$$\|B_2(z,k)\|_{C^\rho([0,T];X)} \leq C(T_0)T^{(1-\gamma)(1+\gamma-\rho)}\|z\|_{C^{\rho-\gamma}([0,T];X)}\|k\|_{C^\gamma([0,T];\mathbf{R})},$$

$$\|L_2 k\|_{C^\rho([0,T];X)} \leq C(T_0)T^{(1-\gamma)(1+\gamma-\rho)}\|k\|_{C^\gamma([0,T];\mathbf{R})}. \square$$

LEMMA 6.3 *Under assumption H9 the linear operator*

$$L_3 k(t) = \int_0^t k(s)A(t)U(t,s)B(0)u_0\,ds \tag{6.19}$$

maps $C([0,T];\mathbf{R})$ into $C^\gamma([0,T];X)$ and satisfies the following estimates:

$$\|L_3 k\|_{C^\gamma([0,T];X)} \leq C(T_0)T^{\beta-\theta-\gamma}\|k\|_{C([0,T];\mathbf{R})}\|B(0)u_0\|_{(D_0,X)_{\theta,p}}, \tag{6.20}$$

D_0 and $(D_0,X)_{\theta,p}$ being defined by (1.10) and (1.14), respectively.

Finally, our last lemma ensures that functions z_0 and k_0 defined by (6.6) and (6.13), respectively, belong to $C^\gamma([0,T];X)$.

LEMMA 6.4 *Let assumptions H1–H9 hold and let $w_0 = L(0)u_0 - f(0) \in D_0$, $A(0)w_0 - f'(0) + L'(0)u_0 \in (D_0,X)_{\beta-\gamma,p}$. Then $z_0 \in C^\gamma([0,T];X)$ and* *satisfies the estimate*

$$\|z_0\|_{C^\gamma([0,T];X)} \le C(T_0)\big[\|w_0\|_{D_0} + \|A(0)w_0 + L'(0)u_0 - f'(0)\|_{(D_0,X)_{\beta-\gamma,p}} + T^{\rho+\beta-1-\gamma}\|f' - L'u_0\|_{C^\rho([0,T];\mathbf{R})}\big].$$

LEMMA 6.5 *Under assumptions H1–H9 function k_0 defined by formula (6.13) actually belongs to $C^\gamma([0,T];X)$ and satisfies the estimate*

$$\|k_0\|_{C^\gamma([0,T];\mathbf{R})} \le C(T_0)\big[\|u_0\| + \|w_0\|_{D_0} + \|A(0)w_0 + L'(0)u_0 - f'(0)\|_{(D_0,X)_{\beta-\gamma,p}} + \|g''\|_{C^\rho([0,T];\mathbf{R})} + T^{\rho+\beta-1-\gamma}\|f' - L'u_0\|_{C^\rho([0,T];X)} + T^{\rho-\gamma}\|f'\|_{C^\rho([0,T];X)}\big].$$

Proof of Lemma 6.4. It immediately follows from Theorem 5.1. replacing (f, v_0) with $(f' - L'u_0, L(0)u_0 - f(0))$. □

Proof of Lemma 6.5. It immediately follows from formula (6.13) and Lemma 6.4. □

Proof of Theorem 1.1. First we estimate in $C^\gamma([0,T];X)$ and $C^\gamma([0,T];\mathbf{R})$, respectively, the nonlinear operators N_1 and N_2 defined in (6.5) and (6.14) and their increments with respect to $(z,k) \in C^{\rho-\gamma}([0,T];X) \times C^\gamma([0,T];\mathbf{R})$ (recall that $\gamma < \rho$ owing to *H7*).

Set now

$$X_1 = X, \quad X_2 = \mathbf{R}. \tag{6.21}$$

From definitions (6.5), (6.14), assumption *H7*, implying $\rho - \gamma \le \gamma$, from Theorem 5.1, with $f(0) = u_0 = 0$, and from Lemmata 6.1, 6.2 we easily deduce the following estimates:

$$\|QL_1 z\|_{C^\gamma([0,T];X)} \le C(T_0)T^{\rho+\beta-1-\gamma}\|L_1 z\|_{C^\rho([0,T];X)}$$

$$\le C(T_0)T^{\beta-\gamma}\|z\|_{C([0,T];X)},$$

$$\|QB_2(z,k)\|_{C^\gamma([0,T];X)} \le C(T_0)T^{\rho+\beta-1-\gamma}\|B_2(z,k)\|_{C^\rho([0,T];X)}$$

$$\le C(T_0)T^{\rho+\beta-1-\gamma+(1-\gamma)(1+\gamma-\rho)}\|z\|_{C^{\rho-\gamma}([0,T];X)}\|k\|_{C^\gamma([0,T];\mathbf{R})}$$

$$\le C(T_0)T^{\beta-\gamma}\|z\|_{C^\gamma([0,T];X)}\|k\|_{C^\gamma([0,T];\mathbf{R})},$$

since $\rho + \beta - 1 - \gamma + (1-\gamma)(1+\gamma-\rho) > \beta - \gamma$.

Analogously

$$\|QL_2 k\|_{C^\gamma([0,T];X)} \le C(T_0)T^{\rho+\beta-1-\gamma}\|L_2 k\|_{C^\rho([0,T];X)}$$

$$\le C(T_0)T^{\rho+\beta-1-\gamma+(1-\gamma)(1+\gamma-\rho)}\|k\|_{C^\gamma([0,T];\mathbf{R})} \le C(T_0)T^{\beta-\gamma}\|k\|_{C^\gamma([0,T];\mathbf{R})}.$$

In conclusion, if $\gamma \in (0, \min\{2\beta - 1, \rho + \beta - 1\})$ and $\theta \in (0, \beta - \gamma)$, we get

$$\|N_1(z,k)\|_{C^\gamma([0,T];X)} \le C(T_0)\|k\|_{C([0,T];\mathbf{R})}T^{\beta-\theta-\gamma}\|B(0)u_0\|_{(D_0;X)_{\theta,p}}$$
$$+C(T_0)T^{\beta-\gamma}\|k\|_{C^\gamma([0,T];\mathbf{R})} + C(T_0)T^{\beta-\gamma}\|z\|_{C([0,T];X)}$$
$$+C(T_0)T^{\beta-\gamma}\|z\|_{C^\gamma([0,T];X)}\|k\|_{C^\gamma([0,T];\mathbf{R})} \le C(T_0)T^{\beta-\theta-\gamma}$$
$$\times\left\{\|z\|_{C([0,T];X)} + \|k\|_{C^\gamma([0,T];\mathbf{R})} + \|z\|_{C^\gamma([0,T];X)}\|k\|_{C^\gamma([0,T];\mathbf{R})}\right\}. \qquad (6.22)$$

To estimate N_2 take first the following estimate into account:

$$\|D_t m\, L^{-1}N_1(z,k)\|_{C^\gamma([0,T];X)} \le C(T_0)\|N_1(z,k)\|_{C^\gamma([0,T];X)}, \qquad (6.23)$$
$$\|L_1 z\|_{C^\gamma([0,T];X)} \le C(T_0)T^{1-\gamma}\|z\|_{C([0,T];X)}, \qquad (6.24)$$
$$\|B_2(z,k)\|_{C^\gamma([0,T];X)} \le C(T_0)T^{1-\gamma}\|k\|_{C^\gamma([0,T];\mathbf{R})}\|z\|_{C([0,T];X)}. \qquad (6.25)$$

From (6.29), (6.23) and Lemmata 5.7, 5.8 we easily deduce the following estimates for N_2, if $\gamma \in (0, \min\{2\beta - 1, \rho + \beta - 1\})$ and $\theta \in (0, \beta - \gamma)$:

$$\|N_2(z,k)\|_{C^\gamma([0,T];X)} \le C(T_0)\big\{\|L_4 z\|_{C^\gamma([0,T];X)}$$
$$+ \|D_t m \cdot L^{-1}N_1(z,k)\|_{C^\gamma([0,T];X)} + \|N_1(z,k)\|_{C^\gamma([0,T];X)}$$
$$+ \|L_1 z\|_{C^\gamma([0,T];X)} + \|L_2 k\|_{C^\gamma([0,T];X)} + \|B_2(z,k)\|_{C^\gamma([0,T];X)}\big\}$$
$$\le C(T_0)T^{\beta-\theta-\gamma}\big\{\|z\|_{C([0,T];X)} + \|k\|_{C^\gamma([0,T])}$$
$$+ \|z\|_{C^\gamma([0,T];X)}\|k\|_{C^\gamma([0,T];X)}\big\}. \qquad (6.26)$$

Then from definitions (6.5) and (6.7) we deduce the following identities that hold for any $t \in [0,T]$ and any pair $(z_1,k_1),(z_2,k_2) \in C^{\rho-\gamma}([0,T];X) \times C^\gamma([0,T];\mathbf{R})$:

$$N_1(z_2,k_2)(t) - N_1(z_1,k_1)(t) = -L_3(k_2-k_1)(t) - QL_2(k_2-k_1)(t)$$
$$- QL_1(z_2-z_1)(t) - QB_2(z_2-z_1,k_2)(t) - QB_2(z_1,k_2-k_1)(t), \quad (6.27)$$
$$N_2(z_2,k_2)(t) - N_2(z_1,k_1)(t) = \chi\Psi[L_4(z_2-z_1)(t)]$$
$$+ \chi\Psi[(D_t m(t)L(t)^{-1} - 1)(N_1(z_2,k_2) - N_1(z_1,k_1))(t)]$$
$$- \chi\Psi[L_1(z_2-z_1)(t)] - \chi\Psi[L_2(k_2-k_1)(t)]$$
$$- \chi\Psi[B_2(z_2-z_1,k_2)(t)] - \chi\Psi[B_2(z_1,k_2-k_1)(t)]. \qquad (6.28)$$

Hence, from (6.27), (6.28), (6.29), (6.30) we easily obtain the following estimates, where $j = 0, 1$:

$$\|N_j(z_2,k_2) - N_j(z_1,k_1)\|_{C^\gamma([0,T];X_j)}$$
$$\leq T^{\beta-\gamma-\theta}C(T_0)\big(\|z_2 - z_1\|_{C([0,T];X)} + \|k_2 - k_1\|_{C^\gamma([0,T];\mathbf{R})} + \|k_2\|_{C^\gamma([0,T];\mathbf{R})}$$
$$\times \|z_2 - z_1\|_{C^\gamma([0,T];X)} + \|z_1\|_{C^\gamma([0,T];X)}\|k_2 - k_1\|_{C^\gamma([0,T];\mathbf{R})}\big). \tag{6.29}$$

To solve the fixed-point system (6.13) and (6.14) let us now introduce the following Banach space

$$Y = C^\gamma([0,T];X) \times C^\gamma([0,T];\mathbf{R}) \tag{6.30}$$

endowed with the norm

$$\|(z,k)\|_Y = \|z\|_{C^\gamma([0,T];X)} + \|k\|_{C^\gamma([0,T];\mathbf{R})}. \tag{6.31}$$

Then, according to (6.29), the vector operator $N = (N_1, N_2)$ maps Y into itself.

Let us now introduce the family of closed balls

$$E(r) = \{(z,k) \in Y : \|(z,k)\|_Y \leq r\}, \qquad \forall r \in (r_0, +\infty), \tag{6.32}$$

where

$$\|(z_0,k_0)\|_Y \leq r_0, \qquad z_0(t) = A(t)\widetilde{w}(t),\ t \in [0,T]. \tag{6.33}$$

From (6.29), (6.29), (6.32) and (6.33) we easily deduce the estimates

$$\|N(z,k)\|_Y \leq T^{\beta-\gamma-\theta}C(T)(r + r^2) \tag{6.34}$$

$$\|N(z_2,k_2) - N(z_1,k_1)\|_Y$$
$$\leq T^{\beta-\gamma-\theta}C(T_0)(1+r)\big(\|z_2 - z_1\|_{C^\gamma([0,T];X)} + \|k_2 - k_1\|_{C^\gamma([0,T];\mathbf{R})}\big), \tag{6.35}$$

From (6.34) and (6.35) we deduce that operator $\widetilde{N}(z,k) = (z_0,k_0) + N(z,k)$ is a contraction mapping from $E(r)$ into itself whenever the pair (T,r) satisfies the system of inequalities

$$\begin{cases} r_0 + T^{\beta-\gamma-\theta}C(T_0)(r + r^2) \leq r, \\ T^{\beta-\gamma-\theta}C(T_0)(1+r) < 1. \end{cases} \tag{6.36}$$

Observe that system (6.36) is solvable for small enough T and any $r \in (r_0, +\infty)$, since the left sides in (6.36) converge, as $T \to 0+$, to r_0 and 0, respectively. Consequently, system (6.3), (6.5) admits, for such T's, a unique solution $(z,k) \in C^\gamma([0,T];X) \times C^\gamma([0,T];\mathbf{R})$. Then function w defined by (6.4) belongs to $C^\gamma([0,T];D_0)$ and solves the direct problem (1.6)–(1.8). Since the right-hand side in (1.6) belongs to $C^\gamma([0,T];X)$, from the regularity results proved in Section 5 for the Cauchy problem (3.28) we conclude that

$C^{1+\gamma}([0,T];X)\cap C^{\gamma}([0,T];D_0)$. Since problem (1.6)–(1.8), is equivalent to the direct problem (1.1)–(1.3) via formula (1.5), we conclude that function u defined by (1.5) belongs to $C^{1+\gamma}([0,T];X)\cap C^{\gamma}([0,T];D_0)$ and solves problem (1.1)–(1.3).

Finally, since problem (1.1)–(1.4) is equivalent to the fixed-point system (6.3) (6.5), we deduce that the pair (u,k) solves our identification problem (1.1)–(1.4). $\square$

We conclude this Section by proving Lemmata 6.1, 6.2, 6.3.

Proof of Lemma 6.1. The estimates for operators L_1 and L_4 can be deduced from the corresponding ones for operators

$$\widetilde{L}_j z(t) = H_j(t)\int_0^t L(s)^{-1}z(s)\,ds, \qquad \forall t\in[0,T],\ j=1,4, \tag{6.37}$$

where H_j, $j=1,4$, satisfies the following inequalities, where $\delta_1=\rho$ and $\delta_2=\gamma$:

$$\|H_j(t)\|\le C,\quad 0\le s\le t\le T, \tag{6.38}$$

$$\|H_j(t_2)-H_j(t_1)\|\le C(t_2-t_1)^{\delta_j},\quad 0\le s\le t_1\le t_2\le T. \tag{6.39}$$

From the inequality

$$\|\widetilde{L}_j z(t)\|\le CT\|z\|_{C([0,T];X)}, \qquad \forall t\in[0,T], \tag{6.40}$$

we immediately deduce

$$\|\widetilde{L}_j z\|_{C([0,T];X)}\le CT\|z\|_{C([0,T];X)}. \tag{6.41}$$

Likewise, the identity

$$\begin{aligned}&\widetilde{L}_j z(t_2)-\widetilde{L}_j z(t_1)\\ &= H_j(t_2)\int_{t_1}^{t_2} L(s)^{-1}z(s)\,ds + [H_j(t_2)-H_j(t_1)]\int_0^{t_1} L(s)^{-1}z(s)\,ds\end{aligned} \tag{6.42}$$

implies the inequalities

$$\begin{aligned}&\|\widetilde{L}_j z(t_2)-\widetilde{L}_j z(t_1)\|\\ &\le C(t_2-t_1)\|z\|_{C([0,T];X)} + C(t_2-t_1)^{\delta_j}T\|z\|_{C([0,T];X)}\\ &\le C(t_2-t_1)^{\delta_j}T^{1-\delta_j}(1+T_j^{\delta})\|z\|_{C([0,T];X)}, \qquad 0\le t_1\le t_2\le T,\end{aligned} \tag{6.43}$$

that concludes the proof of the lemma. $\square$

To prove quickly Lemma 6.2 we premise the following Lemma 6.6 concerning convolutions.

LEMMA 6.6 *For any $g \in C^{\gamma}([0,T];\mathbf{R})$ and $f \in C^{\nu}([0,T];X)$ with $\gamma,\nu \in (0,1)$, $\gamma+\nu \neq 1$, the following estimates hold:*

$$\|g * f\|_{C^{\gamma+\nu}([0,T];X)} \le T^{(1-\gamma)(1-\nu)}(1+T^{\nu})^{1-\gamma}(1+T+T^{1-\nu})^{\gamma}\|f\|_{C^{\nu}([0,T];X)}\|g\|_{C^{\gamma}([0,T];\mathbf{R})}. \quad (6.44)$$

Proof of Lemma 6.6. Associate with any fixed $f \in C^{\nu}([0,T];X)$ the linear operator $M_f(g) = g * f$. Observe now that the following formulae hold for $j = 0,1$ and $g \in C^{j}([0,T];\mathbf{R})$, where $\nu_{h,k}$ denotes the Kronecker delta:

$$D_t^j(g * f)(t) = \nu_{1,j}g(0)f(t) + (D_t^j g) * f(t), \qquad t \in [0,T], \quad (6.45)$$

$$D_t^j(g * f)(t_2) - D_t^j(g * f)(t_1)$$
$$= \nu_{1,j}g(0)[f(t_2) - f(t_1)] + \int_{t_1}^{t_2} D_t^j g(s)f(t_2 - s)\,ds$$
$$+ \int_0^{t_1} D_t^j(g(s))[f(t_2 - s) - f(t_1 - s)]\,ds,\ 0 \le t_1 \le t_2 \le T. \quad (6.46)$$

From (6.45) and (6.46) we easily deduce the following estimates that hold for $g \in C([0,T];\mathbf{R})$ and $g \in C^1([0,T];\mathbf{R})$, respectively:

$$\|M_f(g)\|_{C^{\nu}([0,T];X)} \le (T + T^{1-\nu})\|g\|_{C([0,T];\mathbf{R})}\|f\|_{C^{\nu}([0,T];X)}, \quad (6.47)$$

$$\|M_f(g)\|_{C^{1+\nu}([0,T];X)} \le T\|g\|_{C([0,T];\mathbf{R})}\|f\|_{C^{\nu}([0,T];X)} + |g(0)|\|f\|_{C^{\nu}([0,T];X)}$$
$$+(T + T^{1-\nu})\|g'\|_{C([0,T];\mathbf{R})}\|f\|_{C^{\nu}([0,T];X)}$$
$$\le (1 + T + T^{1-\nu})\|g\|_{C^1([0,T];\mathbf{R})}\|f\|_{C^{\nu}([0,T];X)}. \quad (6.48)$$

Consequently, we have shown that M_f maps continuously $C([0,T];\mathbf{R})$ and $C^1([0,T];\mathbf{R})$ into $C^{\nu}([0,T];X)$ and $C^{1+\nu}([0,T];X)$, respectively.

Using interpolation, we easily conclude that the convolution operator M_f, with a fixed $f \in C^{\nu}([0,T];X)$, maps continuously $C^{\gamma}([0,T];\mathbf{R})$ into $C^{\gamma+\nu}([0,T];X)$ and satisfies estimate (6.44). □

Proof of Lemma 6.2. Lemmata 6.1 and 6.6, with $\nu = \rho - \gamma$, imply that $B_2(k,z), L_2(k) \in C^{\rho}([0,T];X)$ for any $(z,k) \in C^{\rho-\gamma}([0,T];X)\times C^{\gamma}([0,T];X)$, since *H7* implies $\rho > \gamma$ and $B'(\cdot)u_0 \in C^{\rho-\gamma}([0,T];X)$.
Moreover, from Lemma 6.6 we deduce the following estimates that conclude the proof:

$$\|B_2(z,k)\|_{C^{\rho}([0,T];X)} \le C(T_0)T^{(1-\gamma)(1-\nu)}\|z\|_{C^{\rho-\gamma}([0,T];X)}\|k\|_{C^{\gamma}([0,T];\mathbf{R})},$$
$$\|L_2 k\|_{C^{\rho}([0,T];X)} \le C(T_0)T^{(1-\gamma)(1-\nu)}\|k\|_{C^{\gamma}([0,T];\mathbf{R})}, \quad (6.49)$$

C being a positive function continuous up to $T = 0$. □

Proof of Lemma 6.3. Recall that assumptions *H7* and *H9* imply $\gamma < 2\beta - 1$, $\beta > \theta + \gamma$ and $w_1 = B(0)u_0 \in (D_0; X)_{\theta,p}$.

Consider first the following estimates, where we make use of estimate (3.8), with $j = 2$, in Lemma 3.1:

$$\|A(s)\big[e^{-(t_2-s)A(s)} - e^{-(t_1-s)A(s)}\big]w_1\| = \Big\|A(s)\int_{t_1}^{t_2} D_r e^{-(r-s)A(s)} w_1 dr\Big\|$$

$$= \Big\|\int_{t_1}^{t_2} D_r^2 e^{-(r-s)A(s)} B(0)u_0 dr\Big\| \le C(T_0)\int_{t_1}^{t_2} (r-s)^{\beta-\theta-2}\|w_1\|_{(D_0,X)_{\theta,p}} dr$$

$$= \frac{C(T_0)}{1-\beta+\theta}\{(t_1-s)^{\beta-\theta-1} - (t_2-s)^{\beta-\theta-1}\}\|w_1\|_{(D_0,X)_{\theta,p}}$$

$$= C_1(T_0)(t_1-s)^{\beta-\theta-1}\Big\{1 - \Big(\frac{t_1-s}{t_2-s}\Big)^{1-\beta+\theta}\Big\}\|w_1\|_{(D_0,X)_{\theta,p}}$$

$$\le C_1(T_0)(t_1-s)^{\beta-\theta-1}\Big(1 - \frac{t_1-s}{t_2-s}\Big)\|w_1\|_{(D_0,X)_{\theta,p}}$$

$$= C_1(T_0)(t_1-s)^{\beta-\theta-1}\frac{t_2-t_1}{t_2-s}\|w_1\|_{(D_0,X)_{\theta,p}}, \qquad 0 \le t_1 < t_2 \le T_0, \qquad (6.50)$$

since $\theta < \beta$ implies $0 < 1 - \beta + \theta < 1$.

Reasoning as in the proof of Lemma 2.3, we obtain the estimate

$$\|A(t)W(t,s)w_1\| \le C(T_0)(t-s)^{2\beta-\theta-1}\|w_1\|_{(D_0,X)_{\theta,p}}, \qquad (6.51)$$

operator W being defined by (2.28).

Likewise, taking advantage of (6.50), we deduce that the estimate

$$\|\,(A(t_2)W(t_2,s) - A(t_1)W(t_1,s))\,w_1\| \le C(T_0)(t_2-t_1)^{\gamma}\|w_1\|_{(D_0,X)_{\theta,p}}$$
$$\times\big\{(t_2-s)^{2\beta-\theta-1-\gamma} + (t_1-s)^{2\beta-\theta-1-\gamma}\big\} \quad (6.52)$$

holds if $\gamma < 2\beta - 1$, $\theta < \beta$. Indeed, the inequality $\|(A(t)-A(s))A(r)^{-1}\| \le C(T_0)|t-s|^{\rho}$, (cf. (2.10)) holds with $\rho = 1$, since $A(\cdot)A(s)^{-1} \in C^1([0,T];\mathcal{L}(X))$.

From identity (2.29), estimates (6.51), (6.52) and the inclusion $(D_0;X)_{\theta,p} \hookrightarrow (D_0;X)_{\theta,\infty}$ we get

$$\|A(t)U(t,s)w_1\| \le \|A(t)A(s)^{-1}A(s)e^{-(t-s)A(s)}w_1\| + \|A(t)W(t,s)w_1\|$$

$$\le C(T_0)(t-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}} + C(T_0)(t-s)^{2\beta-\theta-1}\|w_1\|_{(D_0,X)_{\theta,p}}$$

$$\le C(T_0)(t-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}},$$

$$\|A(t_2)U(t_2,s)w_1 - A(t_1)U(t_1,s)w_1\|$$

$$\le \|[A(t_2)-A(t_1)]A(s)^{-1}A(s)e^{-(t_2-s)A(s)}w_1\|$$

$$+\,\|A(t_1)A(s)^{-1}A(s)\big[e^{-(t_2-s)A(s))} - e^{-(t_1-s)A(s)}\big]w_1\|$$

$$+\,\|A(t_2)W(t_2,s)w_1 - A(t_1)W(t_1,s)w_1\|$$

$$\leq C(T_0)(t_2-t_1)(t_2-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}}$$
$$+ C(T_0)\frac{t_2-t_1}{t_2-s}(t_1-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}}$$
$$+ C(T_0)(t_2-t_1)^{\gamma}\left\{(t_2-s)^{2\beta-\theta-1-\gamma}+(t_1-s)^{2\beta-\theta-1-\gamma}\right\}\|w_1\|_{(D_0,X)_{\theta,p}}$$
$$\leq C(T_0)(t_2-t_1)^{\gamma}(t_2-s)^{\beta-\theta-\gamma}\|w_1\|_{(D_0;X)_{\theta,p}}$$
$$+ C(T_0)(t_2-t_1)^{\gamma}(t_1-s)^{\beta-\theta-1-\gamma}\|w_1\|_{(D_0;X)_{\theta,p}}$$
$$+ C(T_0)(t_2-t_1)^{\gamma}\left\{(t_2-s)^{2\beta-\theta-1-\gamma}+(t_1-s)^{2\beta-\theta-1-\gamma}\right\}\|w_1\|_{(D_0,X)_{\theta,p}}$$
$$\leq C(T_0)(t_2-t_1)^{\gamma}(t_1-s)^{\beta-\theta-1-\gamma}\|w_1\|_{(D_0;X)_{\theta,p}},$$

since $(t_j-s)^{2\beta-\theta-1-\gamma} \leq T_0(t_j-s)^{\beta-\theta-1-\gamma} \leq T_0(t_1-s)^{\beta-\theta-1-\gamma}$, $j=1,2$. Therefore

$$\|L_3k(t)\| = \left\|\int_0^t k(s)A(t)U(t,s)w_1\,ds\right\|$$
$$\leq C(T_0)\int_0^t |k(s)|(t-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}}\,ds$$
$$\leq C(T_0)\|k\|_{C([0,T])}T^{\beta-\theta}\|w_1\|_{(D_0;X)_{\theta,p}}.$$

2Analogously

$$\|L_3k(t_2)-L_3k(t_1)\| \leq \int_{t_1}^{t_2}|k(s)|\|A(t_2)U(t_2,s)w_1\|\,ds$$
$$+\int_0^{t_1}|k(s)|\|A(t_2)U(t_2,s)-A(t_1)U(t_1,s)]w_1\|\,ds$$
$$\leq C(T_0)\int_{t_1}^{t_2}|k(s)|(t_2-s)^{\beta-\theta-1}\|w_1\|_{(D_0;X)_{\theta,p}}ds$$
$$+ C(T_0)\int_0^{t_1}|k(s)|(t_2-t_1)^{\gamma}(t_1-s)^{\beta-\theta-1-\gamma}\|w_1\|_{(D_0;X)_{\theta,p}}ds$$
$$\leq C(T_0)\|k\|_{C([0,T])}\|w_1\|_{(D_0;X)_{\theta,p}}\left[(t_2-t_1)^{\beta-\theta}+(t_2-t_1)^{\gamma}t_1^{\beta-\theta-\gamma}\right]$$
$$\leq C(T_0)(t_2-t_1)^{\gamma}\|k\|_{C([0,T])}\|w_1\|_{(D_0;X)_{\theta,p}}T^{\beta-\theta-\gamma}.$$

Hence

$$|L_3k|_{C^{\gamma}([0,T];X)} \leq C(T_0)\|k\|_{C([0,T])}T^{\beta-\theta-\gamma}\|w_1\|_{(D_0;X)_{\theta,p}}.$$

The proof is complete. □

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Existence results for a phase transition model based on microscopic movements

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Abstract This note deals with a nonlinear system of PDEs accounting for phase transition phenomena with viscosity terms. The existence of solutions to a Cauchy-Neumann problem is established in the one dimensional space setting, using a regularization – *a priori* estimates – passage to limit procedure. An asymptotic analysis is performed when the viscosity coefficient tends to 0, recovering a – previously investigated – nonlinear system modelling phase changes.

1 Introduction and preliminaries

In this paper we study a Cauchy-Neumann problem related to the following system

$$\partial_t \theta + \theta \partial_t \chi - \partial_{xx}\theta = (\partial_t \chi)^2 + k(\partial_{xt}\chi)^2 \quad \text{in } Q\,, \tag{1.1}$$

$$\partial_t \chi - k\partial_{xxt}\chi - \partial_{xx}\chi + \beta(\chi) \ni \theta - \theta_c \quad \text{in } Q\,, \tag{1.2}$$

for $Q :=]0,\ell[\times]0,T[$, $\ell, T > 0$, where k and θ_c are positive constants, and β is a maximal monotone graph in $\mathbf{R} \times \mathbf{R}$.

The above system can describe a one-dimensional phase transition process with strong dissipation. We refer to the model proposed by Frémond [9], where the thermal evolution of a two-phase material is ruled by two state variables, i.e., the absolute temperature θ and the order parameter χ. The main feature of such a model relies on the consideration that the microscopic movements of the particles give rise to macroscopic effects and this *ansatz* is taken into account in the structure of the energy balance (in the present investigation (1.1) above) which turns out to be highly nonlinear. Moreover, in [9] the field χ plays the role of a phase proportion, hence $0 \le \chi \le 1$, where $\chi = 0$ (resp. $\chi = 1$) stands for, e.g., the pure solid (resp. liquid) phase and $0 < \chi < 1$ denotes the presence of a mixture. The device chosen to represent the constraint is the introduction of the indicator function $I_{[0,1]}$ of the interval $[0,1]$; it yields the presence of its subdifferential $\partial I_{[0,1]}$ which is, in particular, a maximal monotone graph on $\mathbf{R} \times \mathbf{R}$ (see the term β in (1.2) above). Although not addressed in the present contribution, we remark that Frémond's model can also deal with the irreversible evolution of the phase χ; from the mathematical point of view it means the presence of a further maximal monotone graph in (1.2) acting on $\partial_t \chi$ (e.g., $\partial I_{[0,+\infty[}$).

We are going to complement the system (1.1)–(1.2) with the boundary conditions

$$\begin{aligned} &\partial_x \theta(0,\cdot) = \partial_x \theta(\ell,\cdot) = 0, \quad \partial_x \chi(0,\cdot) = \partial_x \chi(\ell,\cdot) = 0, \\ &k\partial_{xt}\chi(0,\cdot) = k\partial_{xt}\chi(\ell,\cdot) = 0 \quad \text{a.e. in } (0,T), \end{aligned} \tag{1.3}$$

and with the initial conditions

$$\theta(\cdot,0) = \theta_0 \qquad \chi(\cdot,0) = \chi_0 \quad \text{in } \Omega\,. \tag{1.4}$$

We point out that most of the physical constants in the reference system (1.1)–(1.2) have been normalized to 1; we only kept the dissipation coefficient k and the temperature of phase transition θ_c. On the other hand, in view of different dynamics, we will be able to deal with a quite general maximal monotone graph β in place of $\partial I_{[0,1]}$.

As for the well posedness of Cauchy-Neumann problems related to Frémond's system of phase transition, *global in time* existence of solutions has been proved only under some restrictions on data or equations: no diffusion in [8] and [14], some kind of small perturbations assumption in [5], [6], [12], an *a priori* maximum speed of phase change in [13], or assuming one dimensional setting [10], [11], [17]. The "full problem" (i.e., without the aforementioned restrictions and simplifications) has only been shown to be *locally* solvable in time [15]. Of course, the highly nonlinear structure and, in particular, the quadratic terms in (1.1) are responsible for this drawback.

The aim of this paper is twofold. First, to prove a global existence result for the problem in one space dimension. Second, to connect the latter existence result with the (reversible) nondissipation case of [11] by means of an asymptotic analysis as $k \downarrow 0$.

The plan of the paper is as follows. In the rest of this Section we provide the general setting and state the main results. In the next Section we consider a family of regularized problems and prove their *local* well posedness through a fixed point procedure of Schauder type. The *a priori* estimates of the subsequent Section allow the extension to the whole time interval. Moreover, since they hold independently of the regularization parameter, we can deduce the proper weak and strong convergences to the solution of the original problem (1.1)–(1.4). Finally, a closely related argument leads to an asymptotic analysis for $k \downarrow 0$, recovering in the limit the problem studied in [11].

We go on fixing some notation. We set

$$\Omega :=]0,\ell[, \quad Q_t :=]0,\ell[\times]0,t[\quad \forall t \in]0,T], \quad Q := Q_T.$$

Next, we let

$$H := L^2(\Omega), \quad V := H^1(\Omega),$$
$$W := \{u \in H^2(\Omega) \text{ such that } u'(0) = u'(\ell) = 0\}, \tag{1.5}$$

and we identify H with its dual space H', so that

$$V \subset H \subset V',$$

with dense, compact and continuous embeddings. Besides, we let the symbol $\|\cdot\|$ denote the standard norm of H, while $\|\cdot\|_E$ stands for the norm of the generic normed space E. Moreover, we denote by $<\cdot,\cdot>$ the duality pairing between V' and V, by $(\cdot,\cdot)$ the scalar product in H, and by $J : V \to V'$ the Riesz isomorphism of V onto V'.

We note that, thanks to the one dimensional framework of our problems, we have the continuous injections

$$L^1(\Omega) \subset V', \qquad V \subset L^\infty(\Omega). \tag{1.6}$$

Hence, there exist two positive constants c_1 and c_2 such that the following relations hold

$$\|u\|_{V'} \le c_1 \|u\|_{L^1(\Omega)}, \quad \forall u \in L^1(\Omega),$$
$$\|u\|_{L^\infty(\Omega)} \le c_2 \|u\|_V, \quad \forall u \in V. \tag{1.7}$$

Now, we recall an elementary inequality which will be useful in the sequel

$$ab \le (\delta/2)a^2 + (2\delta)^{-1}b^2 \quad \forall a, b \in \mathbf{R}, \quad \delta > 0. \tag{1.8}$$

Finally, we also remark that there exists a positive constant c_3 depending only on T such that the following estimate holds for any $u \in H^1(0,T;H)$

$$\|u\|^2_{L^2(0,t;H)} \le c_3 \left(\|u(0)\|^2 + \int_0^t \|\partial_t u\|^2_{L^2(0,s;H)} ds \right) \quad \forall t \in (0,T]. \tag{1.9}$$

We give here the precise statement of our problem, introducing the following assumptions on the data.

$$\theta_c, \theta^* > 0 \text{ are assigned constants,} \tag{1.10}$$

$$\varphi : \mathbf{R} \to [0, +\infty] \quad \text{is proper, convex and lower semicontinuous,}$$

$$\varphi(0) = 0, \quad \text{and there exist} \quad c_4,\ c_5 > 0 \quad \text{such that}$$

$$\varphi(r) \ge c_4 r^2 - c_5 \quad \forall r \in D(\varphi) \quad \text{and } \beta := \partial\varphi, \tag{1.11}$$

$$\theta_0 \in V \text{ and } \theta_0 \ge \theta^* \quad \text{in } \overline{\Omega}, \tag{1.12}$$

$$\chi_0 \in W, \tag{1.13}$$

$$\chi_0 \in D(\beta) \text{ a.e. in Q, and } \beta^0(\chi_0) \in H, \tag{1.14}$$

where $D(\varphi)$ and $D(\beta)$ denote the effective domains of φ and β, respectively, ∂ represents as usual the subdifferential in the sense of Convex Analysis and $\beta^0(\chi_0)$ stands for the element of minimal norm of the set $\beta(\chi_0)$ (cf. [7, p. 28]).

Let us now introduce the functionals

$$\Phi(u) := \begin{cases} \int_\Omega \varphi(u(x))\,dx & \text{if } u \in H \text{ and } \varphi(u) \in L^1(\Omega)\,, \\ +\infty & \text{if } u \in H \text{ and } \varphi(u) \notin L^1(\Omega)\,, \end{cases} \tag{1.15}$$

$$\Phi_V(u) := \Phi(v)\,, \quad \forall v \in V\,. \tag{1.16}$$

Moreover we denote by $\beta_{V,V'} := \partial\Phi_V : V \to 2^{V'}$ the corresponding subdifferential. On the other hand, we readily have that [3, Prop. 2.8, p. 61]

$$v \in \partial\Phi(u) \Longleftrightarrow u \in H \quad \text{and } v \in \beta(u) \text{ a.e. in } \Omega \tag{1.17}$$

so that we will use the same symbol β for $\partial\Phi$ with no ambiguity.

Hence, we are in a position to state the following

Problem (P). *Find a triplet (θ, χ, η) such that*

$$\theta \in H^1(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V), \tag{1.18}$$

$$\chi \in W^{1,\infty}(0,T;V), \tag{1.19}$$

$$\eta \in L^\infty(0,T;V'), \tag{1.20}$$

$$< \partial_t\theta \,+\, \theta\,\partial_t\chi, v > \,+ (\partial_x\theta, \partial_x v) =< (\partial_t\chi)^2 + k(\partial_{xt}\chi)^2, v > \qquad \forall v \in V \quad \text{a.e. in }]0,T[, \tag{1.21}$$

$$< \partial_t\chi + \eta, v > \,+ (k\partial_{xt}\chi + \partial_x\chi, \partial_x v) =< \theta - \theta_c, v > \qquad \forall v \in V \quad \text{a.e. in }]0,T[, \tag{1.22}$$

$$\eta \in \beta_{V,V'}(\chi) \quad \text{a.e. in }]0,T[, \tag{1.23}$$

$$\exists\theta_* > 0 \quad \text{such that } \theta \geq \theta_* \quad \text{a.e. in } Q_T, \tag{1.24}$$

$$\theta(\cdot,0) = \theta_0, \quad \chi(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{1.25}$$

REMARK 1.1 Let us stress that the coercivity assumption on φ in (1.11) is perfectly motivated in our framework since $I_{[0,1]}(r) \geq r^2 - 1$ for all $r \in [0,1]$.

Now, we are able to state the main result of the paper.

THEOREM 1.1 *Let assumptions (1.10)–(1.14) hold. Then Problem (P) admits at least a solution.*

REMARK 1.2 In order to prove Thm. 1.1 one could indeed weaken (1.12) by requiring $\theta_0 \in H$ only. Nevertheless, we should need $\theta_0 \in V$ to establish the well-posedness of the approximating problems and to carry out the asymptotic analysis. Hence, for the sake of simplicity we assume $\theta_0 \in V$ in the whole paper instead of considering some suitable approximation.

The proof of this result will be carried out throughout the remainder of the paper by exploiting an approximation procedure. Indeed, we replace β with its Yosida approximation β_ε and solve locally (in time) the regularized problem by the means of fixed point techniques. Then, global *a priori* estimates independent of ε are established and the passage to the limit is obtained via compactness and monotonicity arguments.

As mentioned above, we will be in a position to prove the convergence as $k \downarrow 0$ of the global solution to Problem (P) to a corresponding suitable solution to the same problem with $k = 0$. For the sake of clarity, we shall state this convergence result as follows.

THEOREM 1.2 *Let* $(\theta^k, \chi^k, \eta^k)$ *be a solution to the Problem* (P)*. Then there exists a triplet* $(\theta^0, \chi^0, \eta^0)$ *such that*

$$\theta^0 \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{1.26}$$

$$\chi^0 \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;W), \tag{1.27}$$

$$\eta^0 \in L^\infty(0,T;H), \tag{1.28}$$

$$\partial_t\theta^0 + \theta^0\,\partial_t\chi^0 - \partial_{xx}\theta^0 = (\partial_t\chi^0)^2 \quad a.e.\ in\ Q, \tag{1.29}$$

$$\partial_t\chi^0 + \eta^0 - \partial_{xx}\chi^0 = \theta^0 - \theta_c \quad a.e.\ in\ Q, \tag{1.30}$$

$$\eta^0 \in \beta(\chi^0) \quad a.e.\ in\ Q, \tag{1.31}$$

$$\exists \theta_* > 0 \quad such\ that \quad \theta^0 \geq \theta_* \quad a.e.\ in\ \ Q, \tag{1.32}$$

$$\theta^0(\cdot,0) = \theta_0, \quad \chi^0(\cdot,0) = \chi_0 \quad a.e.\ in\ \ \Omega \tag{1.33}$$

and the following convergences hold

$$\theta_k \rightharpoonup^* \theta^0 \quad in\ \ H^1(0,T;V') \cap L^\infty(0,T;H) \cap L^2(0,T;V), \tag{1.34}$$

$$\chi_k \rightharpoonup^* \chi^0 \quad in\ W^{1,\infty}(0,T;H) \cap H^1(0,T;V), \tag{1.35}$$

$$k^{1/2}\chi_k \rightharpoonup^* 0 \quad in\ W^{1,\infty}(0,T;V) \cap L^\infty(0,T;W), \tag{1.36}$$

$$\eta_k \rightharpoonup^* \eta^0 \quad in\ \ L^\infty(0,T;V'). \tag{1.37}$$

2 Approximation

In order to prove Theorem 1.1, we apply a regularization procedure to the maximal monotone graph β. Namely, we let β_ε be the Yosida approximation of β (we refer to [7] for details) and, consequently, denote by φ_ε the unique primitive of β_ε verifying $\varphi_\varepsilon(0) = 0$. We note that (see [7, p. 28]) one has

$$|\beta_\varepsilon(r)| \leq |\beta^0(r)| \quad \text{for all } \varepsilon > 0 \ \text{ and } \quad r \in D(\beta). \tag{2.1}$$

Moreover, it is well known that φ_ε is given by

$$\varphi_\varepsilon(r) = \min_{s \in D(\varphi)} \left(\frac{1}{2\varepsilon}|r-s|^2 + \varphi(s) \right). \tag{2.2}$$

Thus, we readily have that

$$\varphi_\varepsilon(r) \leq \varphi(r)\,, \quad \forall r \in D(\varphi). \tag{2.3}$$

Moreover, the function φ_ε is defined in all of $\mathbf{R}$ and, taking into account the coercivity assumption in (1.11), it turns out to be coercive as well. Namely, we have

$$\varphi_\varepsilon(r) \geq \frac{c_4}{2} r^2 - c_5\,, \quad \forall r \in \mathbf{R},\ \forall \varepsilon \in (0, (2c_4)^{-1}). \tag{2.4}$$

Indeed, let us consider $r \in \mathbf{R}$, $s \in D(\varphi)$ and $\varepsilon \in (0, (2c_4)^{-1})$. Then

$$\frac{c_4}{2} r^2 \leq c_4|r-s|^2 + c_4 s^2 \leq \frac{1}{2\varepsilon}|r-s|^2 + c_4 s^2 - c_5 + c_5$$
$$\leq \frac{1}{2\varepsilon}|r-s|^2 + \varphi(s) + c_5,$$

from which (2.4) follows.

Finally, we will use the notation

$$\Phi^\varepsilon(u) := \int_\Omega \varphi_\varepsilon(u(x))\,dx \quad \forall u \in H,$$

$$\Phi^\varepsilon_V(v) := \Phi^\varepsilon(v) \quad \forall v \in V$$

(which are of course convex, proper, and lower semicontinuous on H and V, respectively) and observe that [3, Prop. 2.8, p. 61]

$$v \in \partial\Phi^\varepsilon(u) \Longleftrightarrow u \in H \quad \text{and } v \in \beta_\varepsilon(u) \text{ a.e. in } \Omega. \tag{2.5}$$

Owing to the latter remark we will use the same symbol β_ε for $\partial\Phi^\varepsilon$ without ambiguity. Before going on, let us recall the analysis in [4] and let us observe that

$$\partial\Phi^\varepsilon_V(v) = \beta_\varepsilon(v) \quad \forall v \in V. \tag{2.6}$$

Let us introduce the approximating problems (the regularization parameter $\varepsilon > 0$ being fixed).

Problem ($\mathbf{P}_\varepsilon$). *Find a pair $(\theta_\varepsilon, \chi_\varepsilon)$ such that*

$$\theta_\varepsilon \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W), \tag{2.7}$$

$$\chi_\varepsilon \in H^2(0,T;W), \tag{2.8}$$

$$\partial_t\theta_\varepsilon + \theta_\varepsilon\partial_t\chi_\varepsilon - \partial_{xx}\theta_\varepsilon = (\partial_t\chi_\varepsilon)^2 + k(\partial_{xt}\chi_\varepsilon)^2, \quad \text{a.e. in } \ Q_T \tag{2.9}$$

$$\partial_t\chi_\varepsilon - k\partial_{xxt}\chi_\varepsilon - \partial_{xx}\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon) = \theta_\varepsilon - \theta_c, \quad \text{a.e. in } \ Q_T \tag{2.10}$$

$$\exists\theta_* > 0 \quad \text{independent of } \varepsilon \text{ such that } \theta_\varepsilon > \theta_* \quad \text{a.e. in } \ Q_T, \tag{2.11}$$

$$\theta_\varepsilon(\cdot,0) = \theta_0, \quad \chi_\varepsilon(\cdot,0) = \chi_0 \quad \text{a.e. in } \ \Omega. \tag{2.12}$$

THEOREM 2.1 *Let assumptions (1.10)–(1.14) hold. Then Problem (P_ε) admits one and only one solution.*

Without any loss of generality we will take $k = 1$ in the remainder of this Section.

We start with the proof of the existence part of Theorem 2.1. To this aim, we apply the Schauder theorem to a suitable operator $\mathcal{T}$ that will be constructed below. For the sake of brevity we will not detail the whole procedure.

For $R > 0$, let us denote by $Y(\tau, R)$ the closed ball of $H^1(0,\tau;W^{1,4}(\Omega))$ with center 0 and radius R, i.e.,

$$Y(\tau,R) = \{v \in H^1(0,\tau;W^{1,4}(\Omega)) \text{ such that } \|v\|_{H^1(0,\tau;W^{1,4}(\Omega))} \le R\}\,, \tag{2.13}$$

where $\tau \in]0,T]$ will be determined later in such a way that $\mathcal{T} : Y(\tau,R) \to Y(\tau,R)$ turns out to be a compact and continuous operator.

We consider the following auxiliary problems whose well-posedness is guaranteed by standard arguments (hence, for the sake of brevity, we omit any detail).

Let $\widehat{\chi} \in Y(\tau,R)$ be fixed and let $\overline{\theta} := \mathcal{T}_1(\widehat{\chi})$ be the *unique* solution to the following

Problem 1. Given $\widehat{\chi} \in Y(\tau,R)$, find $\overline{\theta}$ such that

$$\overline{\theta} \in [W^{1,1}(0,\tau;H) + H^1(0,\tau;V')] \cap C^0([0,\tau];H) \cap L^2(0,\tau;V)\,, \tag{2.14}$$

$$< \partial_t\overline{\theta}, v > + (\overline{\theta}\partial_t\widehat{\chi}, v) + (\partial_x\overline{\theta}, \partial_x v) = ((\partial_t\widehat{\chi})^2 + (\partial_{xt}\widehat{\chi})^2, v) \qquad \forall v \in V \quad \text{a.e. in }]0,\tau[\,, \tag{2.15}$$

$$\overline{\theta}(\cdot,0) = \theta_0 \quad \text{a.e. in } \Omega\,. \tag{2.16}$$

Now, given such a $\overline{\theta}$, let $\overline{\chi}$, with $\overline{\chi} := \mathcal{T}_2(\overline{\theta})$, be the *unique* solution of the following

Problem 2. Given $\overline{\theta}$ satisfying the regularity in (2.14), find $\overline{\chi}$ such that

$$\overline{\chi} \in H^1(0,\tau;W)\,, \tag{2.17}$$

$$\partial_t\overline{\chi} - \partial_{xxt}\overline{\chi} - \partial_{xx}\overline{\chi} + \beta_\varepsilon(\overline{\chi}) = \overline{\theta} - \theta_c \quad \text{a.e. in } Q_\tau\,, \tag{2.18}$$

$$\overline{\chi}(\cdot,0) = \chi_0 \quad \text{a.e. in } \Omega\,. \tag{2.19}$$

Finally, we define the operator $\mathcal{T}$ as the composition $\mathcal{T}_2 \circ \mathcal{T}_1$. Our aim is to show that, at least for small times, the Schauder theorem applies to the map $\mathcal{T}$ from $Y(\tau,R)$ into itself. Namely, we will prove that there exists $\tau > 0$ such that $\mathcal{T}$ satisfies the following properties

$\mathcal{T}$ maps $Y(\tau,R)$ into itself;

$\mathcal{T}$ is compact;

$\mathcal{T}$ is continuous.

We start by deriving some *a priori* bounds on $\overline{\theta}$ and $\overline{\chi}$. We warn that in the proofs we employ the same symbol c for different constants (independent of τ and R, but possibly depending on ε), even in the same formula, in regard to simplicity. In particular, we stress that all constants do not blow up as τ becomes small. Now, in order to obtain *a priori* bounds on $\overline{\theta}$, we choose $v = \overline{\theta}$ in (2.15) and integrate from 0 to t, with $0 < t < \tau$. Owing to (1.8), the Hölder inequality and the continuous injection $V \hookrightarrow L^4(\Omega)$, we have

$$\begin{aligned}
&\frac{1}{2}\|\overline{\theta}(t)\|^2 + \|\partial_x\overline{\theta}\|^2_{L^2(0,t;H)} \le \frac{1}{2}\|\theta_0\|^2 \\
&\quad + c\int_0^t \|\overline{\theta}(s)\|_{L^4(\Omega)}\|\partial_t\widehat{\chi}(s)\|_{L^4(\Omega)}\|\overline{\theta}(s)\|\,ds \\
&\quad + \int_0^t \Big(\|\partial_t\widehat{\chi}(s)\|^2_{L^4(\Omega)} + \|\partial_{xt}\widehat{\chi}(s)\|^2_{L^4(\Omega)}\Big)\|\overline{\theta}(s)\|\,ds \\
&\le \frac{1}{2}\|\theta_0\|^2 + c\int_0^t \Big(\|\partial_t\widehat{\chi}(s)\|_{L^4(\Omega)}\|\overline{\theta}(s)\|_V \\
&\quad + \|\partial_t\widehat{\chi}(s)\|^2_{L^4(\Omega)} + \|\partial_{xt}\widehat{\chi}(s)\|^2_{L^4(\Omega)}\Big)\|\overline{\theta}(s)\|\,ds\,.
\end{aligned} \tag{2.20}$$

Next, in order to recover the full V-norm of $\overline{\theta}$ in the left-hand side, we add to (2.20) $\|\overline{\theta}\|^2_{L^2(0,t;H)}$. Then, we use (1.8) and get

$$\begin{aligned}
&\frac{1}{2}\|\overline{\theta}(t)\|^2 + \|\overline{\theta}\|^2_{L^2(0,t;V)} \le \frac{1}{2}\|\theta_0\|^2 + \frac{1}{2}\|\overline{\theta}\|^2_{L^2(0,t;V)} \\
&\quad + c\int_0^t \Big(1 + \|\partial_t\widehat{\chi}(s)\|^2_{L^4(\Omega)}\Big)\|\overline{\theta}(s)\|^2\,ds \\
&\quad + c\int_0^t \Big(\|\partial_t\widehat{\chi}(s)\|^2_{L^4(\Omega)} + \|\partial_{xt}\widehat{\chi}(s)\|^2_{L^4(\Omega)}\Big)\|\overline{\theta}(s)\|\,ds\,.
\end{aligned} \tag{2.21}$$

Recalling that, by the definition of $Y(\tau, R)$, the functions $\|\partial_t \widehat{\chi}\|^2_{L^4(\Omega)}$ and $\left(\|\partial_t \widehat{\chi}\|^2_{L^4(\Omega)} + \|\partial_{xt} \widehat{\chi}\|^2_{L^4(\Omega)}\right)$ belong to $L^1(0,\tau)$, we can apply to (2.21) a generalized version of the Gronwall lemma introduced in [2] and we deduce that there exists a positive constant c depending on T, Ω and R such that

$$\|\overline{\theta}\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \leq c\,. \tag{2.22}$$

Finally, the definition of $Y(\tau, R)$ and a comparison in (2.15) yield the regularity in (2.14).

Next, in order to obtain *a priori* bounds on $\overline{\chi}$, we multiply (2.18) by $\partial_t \overline{\chi}$ and integrate over Q_t. Using the Lipschitz continuity of β_ε, the Hölder inequality and relations (1.8), (1.9), we have

$$\begin{aligned}
&\|\partial_t \overline{\chi}\|^2_{L^2(0,t;H)} + \|\partial_{xt} \overline{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x \overline{\chi}(t)\|^2 \\
&\quad \leq \frac{1}{2}\|\partial_x \chi_0\|^2 + c\int_{Q_t} (|\overline{\chi}| + 1)|\partial_t \overline{\chi}| + \int_{Q_t} |(\overline{\theta} - \theta_c)\partial_t \overline{\chi}| \\
&\quad \leq c + \frac{1}{2}\|\partial_x \chi_0\|^2 + c\int_0^t \|\partial_t \overline{\chi}\|^2_{L^2(0,s;H)} ds \\
&\qquad + c\|\overline{\theta} - \theta_c\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_t \overline{\chi}\|^2_{L^2(0,t;H)}\,.
\end{aligned} \tag{2.23}$$

Thanks to (2.22), we deduce that there exists a positive constant c such that

$$\|\overline{\chi}\|_{H^1(0,\tau;V)} \leq c\,. \tag{2.24}$$

On account of (1.6), from (2.24) it follows

$$\|\overline{\chi}\|_{L^\infty(Q_\tau)} \leq c\,. \tag{2.25}$$

Next, we multiply (2.18) by $-\partial_{xxt}\overline{\chi}$, we integrate over Q_t and, thanks to the Lipschitz continuity of β_ε, we obtain

$$\begin{aligned}
&\|\partial_{xt} \overline{\chi}\|^2_{L^2(0,t;H)} + \|\partial_{xxt} \overline{\chi}\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_{xx} \overline{\chi}(t)\|^2 \\
&\quad \leq \frac{1}{2}\|\partial_{xx} \chi_0\|^2 + \int_{Q_t} |\beta_\varepsilon(\overline{\chi})\partial_{xxt} \overline{\chi}| + \int_{Q_t} |(\overline{\theta} - \theta_c)\partial_{xxt} \overline{\chi}| \\
&\quad \leq \frac{1}{2}\|\partial_{xx} \chi_0\|^2 + c(\|\overline{\chi}\|^2_{L^\infty(\Omega)} + 1) \\
&\qquad + c\|\overline{\theta} - \theta_c\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_{xxt} \overline{\chi}\|^2_{L^2(0,t;H)}\,.
\end{aligned} \tag{2.26}$$

Thus, on account of (2.22) and (2.25) from (2.26) we deduce the further bound

$$\|\overline{\chi}\|_{H^1(0,\tau;W)} \leq c\,. \tag{2.27}$$

Next, taking the H-norm of both sides of (2.18), we get $J\partial_t\overline{\chi} = \partial_{xx}\overline{\chi} - \beta_\varepsilon(\overline{\chi}) + \overline{\theta} - \theta_c$ (J being the Riesz isomorphism between V and V'). Thus, evaluating the H-norm, using (2.24), (2.25), (2.22), and taking the supremum w.r.t. time, we have the bound

$$\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;W)} \leq c\,. \tag{2.28}$$

Finally, we differentiate with respect to t both sides of (2.18); thanks to (2.27), (2.14) and the Lipschitz continuity of β_ε, a comparison in the resulting relations gives the bound

$$\|\partial_{tt}\overline{\chi}\|_{L^1(0,\tau;V)} \leq c\,. \tag{2.29}$$

Now, our aim is to find $\tau > 0$ such that the operator $\mathcal{T} : Y(\tau, R) \to Y(\tau, R)$ turns out to be welldefined. Exploiting the previous estimates (cf. (2.28)), we have

$$\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;W^{1,4}(\Omega))} \leq c\,. \tag{2.30}$$

Thus, by the Hölder inequality, we find

$$\|\overline{\chi}\|_{H^1(0,\tau;W^{1,4}(\Omega))} \leq c\sqrt{\tau}\|\overline{\chi}\|_{W^{1,\infty}(0,\tau;W^{1,4}(\Omega))} \leq c\sqrt{\tau}\,. \tag{2.31}$$

Hence, we can take τ so small that

$$c\sqrt{\tau} \leq R\,, \tag{2.32}$$

which ensures that $\overline{\chi}$ belongs to $Y(\tau, R)$.

Next, we observe that the above arguments (cf. (2.27) and (2.29)) lead to

$$\|\overline{\chi}\|_{W^{2,1}(0,\tau;V)\cap W^{1,\infty}(0,\tau;W)} \leq c\,, \tag{2.33}$$

for some positive constant c independent of the choice of $\widehat{\chi}$ in $Y(\tau, R)$, which guarantees that $\mathcal{T}$ is a compact operator. Hence, in order to apply the Schauder theorem, it remains to show that $\mathcal{T}$ is continuous with respect to the natural topology induced in $Y(\tau, R)$ by $H^1(0, \tau; W^{1,4}(\Omega))$. To this aim, we consider a sequence $\widehat{\chi}_n$ in $Y(\tau, R)$ such that

$$\widehat{\chi}_n \to \widehat{\chi} \quad \text{in } Y(\tau, R)\,, \tag{2.34}$$

as $n \to +\infty$. Now, we denote by $\overline{\theta}_n$ the sequence of the solutions to Problem 1 once $\widehat{\chi}$ is substituted by $\widehat{\chi}_n$, i.e.,

$$\overline{\theta}_n := \mathcal{T}_1(\widehat{\chi}_n). \tag{2.35}$$

Arguing as in the derivation of (2.22), we can find a positive constant c not depending on n such that

$$\|\overline{\theta}_n\|_{L^\infty(0,\tau;H)\cap L^2(0,\tau;V)} \leq c\,. \tag{2.36}$$

By wellknown compactness results, there exists a subsequence of n, still denoted by n for the sake of brevity, such that

$$\overline{\theta}_n \rightharpoonup^* \overline{\theta} \ \text{ in } L^\infty(0,\tau;H) \cap L^2(0,\tau;V)\,, \tag{2.37}$$

as $n \to +\infty$. In order to show that $\overline{\theta}$ in (2.37) is the solution to Problem 1 related to $\widehat{\chi}$, i.e., $\overline{\theta} = \mathcal{T}_1(\widehat{\chi})$, we can pass to the limit in (2.15) written at the step n as $n \to +\infty$. We remark, in particular, that for the nonlinear term we have $\overline{\theta}_n \partial_t \widehat{\chi}_n \rightharpoonup \overline{\theta}\partial_t\widehat{\chi}$ in $L^2(0,\tau;H)$, thanks to (2.34) and (2.37). Moreover, thanks to the uniqueness of solution to Problem 1, we deduce that the whole sequence $\overline{\theta}_n$ converges to $\overline{\theta}$, as $n \to +\infty$.

As a second step we consider the sequence $\overline{\chi}_n$ of the solutions to Problem 2 once $\overline{\theta}$ is substituted by $\overline{\theta}_n$, i.e., we have

$$\overline{\chi}_n := \mathcal{T}_2(\overline{\theta}_n) = \mathcal{T}_2 \circ \mathcal{T}_1(\widehat{\chi}_n) = \mathcal{T}(\widehat{\chi}_n). \tag{2.38}$$

Repeating the estimates (cf. (2.27) and (2.29)), we find a positive constant c independent of n such that

$$\|\overline{\chi}_n\|_{W^{2,1}(0,\tau;V)\cap W^{1,\infty}(0,\tau;W)} \le c\,. \tag{2.39}$$

Hence, there exists a subsequence of n, again not relabeled, such that

$$\overline{\chi}_n \rightharpoonup^* \overline{\chi} \ \text{ in } \ W^{1,\infty}(0,\tau;W) \tag{2.40}$$

as $n \to +\infty$. Moreover, by compactness (see [18, Thm. 4, Cor. 5]) from (2.39), we can deduce that

$$\overline{\chi}_n \to \overline{\chi} \ \text{ in } \ H^1(0,\tau;W^{1,4}(\Omega))\,. \tag{2.41}$$

The above convergences, (2.37) and the Lipschitz continuity of β_ε allow us to pass to the limit in the relation (2.18). Thus, thanks to the uniqueness result holding for Problem 2, we have that the whole sequence $\overline{\chi}_n$ converges to $\mathcal{T}_2(\overline{\theta}) = \mathcal{T}_2 \circ \mathcal{T}_1(\widehat{\chi}) = \mathcal{T}(\widehat{\chi})$ and we can identify $\overline{\chi}$ with $\mathcal{T}(\widehat{\chi})$. Namely, by (2.41), we have proved that

$$\mathcal{T}(\widehat{\chi}_n) \to \mathcal{T}(\widehat{\chi}) \ \text{ in } \ H^1(0,\tau;W^{1,4}(\Omega)) \tag{2.42}$$

which concludes the proof of the continuity of the operator $\mathcal{T}$. Thus, by Schauder's fixed point theorem, $\mathcal{T}$ has a fixed point in $Y(\tau,R)$, i.e., there exists at least a local in time solution of the system (2.9)–(2.12), defined on the interval $]0,\tau[$. Note that, at the moment, (2.9) is satisfied only in a weak sense (cf. (2.15)). However, performing some standard parabolic estimates and taking advantage of (1.12) and (2.17), we can prove the further regularity for θ specified in (2.7) so that (2.9) is satisfied a.e. in Q_τ.

Now, we have to discuss the extension of the latter solution to the whole interval $]0,T[$. This will eventually follow from a set of global in time *a priori* estimates which are detailed in the next Section. The latter will entail, in particular, that the solution is indeed global on $]0,T[$. Hence, without loss of generality we will simplify the forthcoming estimates by assuming at once that we are given solutions to Problem (P_ε) which are defined on all of $]0,T[$.

The lower bound (1.24) for the temperature stems from a straightforward application of [16, Thm. 1]).

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3 A priori estimates and passage to the limit

Since we are also interested in an asymptotic analysis when $k \downarrow 0$, we are going to deduce some *a priori* estimates for $(\theta_\varepsilon, \chi_\varepsilon)$ independent of the regularization parameter ε, paying attention to the role of the dissipation terms. Henceforth, let C denote any constant, possibly depending on the data, but neither on ε nor on k. Of course, C may vary from line to line.

3.1 First estimate

Let us integrate (2.9) over Q_t. Moreover, we multiply (2.10) by $\partial_t \chi_\varepsilon$ and integrate over Q_t. Taking the sum of the resulting expressions and performing some cancellations, we obtain

$$\begin{aligned} &\int_\Omega \theta_\varepsilon(t) + \frac{1}{2}\|\partial_x \chi_\varepsilon(t)\|^2 + \int_\Omega \varphi_\varepsilon(\chi_\varepsilon(t)) \\ &\le \int_\Omega \theta_0 + \frac{1}{2}\|\partial_x \chi_0\|^2 + \int_\Omega \varphi_\varepsilon(\chi_0) + \theta_c \left| \int_\Omega \big(\chi_\varepsilon(t) - \chi_0\big) \right| \\ &\le \int_\Omega \varphi(\chi_0) + \theta_c \|\chi_\varepsilon(t)\|_{L^1(\Omega)} + C, \end{aligned} \tag{3.1}$$

where we have also used assumptions (1.12), (1.13) and property (2.3). In order to control the right hand side of (3.1) it suffices to recall (1.14) and observe that (see (2.4))

$$\theta_c \|\chi_\varepsilon(t)\|_{L^1(\Omega)} \le \frac{c_4}{4}\|\chi_\varepsilon(t)\|^2 + C \le \frac{1}{2}\int_\Omega \varphi_\varepsilon(\chi_\varepsilon(t)) + C,$$

whenever ε is small enough.

Hence, moving from (2.11), we readily deduce that

$$\|\theta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{3.2}$$

$$\|\chi_\varepsilon\|_{L^\infty(0,T;V)} \le C, \tag{3.3}$$

$$\|\varphi_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} \le C, \tag{3.4}$$

at least for sufficiently small ε.

3.2 Second estimate

Let us multiply equation (2.9) by the function $-\theta_\varepsilon^{-1}$. The latter choice turns out to be admissible since, by (2.11), $-\theta_\varepsilon^{-1} \in L^\infty(Q_T)$. Moreover, we integrate on Q_t, and exploit (1.12) and (3.2) in order to get

$$-\int_\Omega \ln\left(\theta_\varepsilon(t)\right) + \int_{Q_t}\left(\frac{(\partial_x\theta_\varepsilon)^2}{\theta_\varepsilon^2} + \frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon} + \frac{k(\partial_{xt}\chi_\varepsilon)^2}{\theta_\varepsilon}\right)$$
$$= -\int_\Omega \ln(\theta_0) + \int_{Q_t}\partial_t\chi_\varepsilon \le -\int_\Omega \ln(\theta_0) + \frac{1}{2}\int_{Q_t}\left(\frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon} + \theta_\varepsilon\right)$$
$$\le C + \frac{1}{2}\int_{Q_t}\frac{\partial_t\chi_\varepsilon^2}{\theta_\varepsilon}.$$

Of course the first term in the above left-hand side is bounded from below by virtue of (3.2). Thus, the bound (3.2) and the continuity of the inclusion $W^{1,1}(\Omega) \subset L^\infty(\Omega)$ entail in particular that

$$\int_0^T \|\theta_\varepsilon\|_{L^\infty(\Omega)} = \int_0^T \|\theta_\varepsilon^{1/2}\|^2_{L^\infty(\Omega)} \le C\int_0^T \left(\|\partial_x(\theta_\varepsilon^{1/2})\|^2_{L^1(\Omega)} + \|\theta_\varepsilon\|^2_{L^1(\Omega)}\right)$$
$$\le C\left(1 + \int_0^T\left(\int_\Omega\left|\frac{\partial_x\theta_\varepsilon}{\theta_\varepsilon^{1/2}}\right|\right)^2\right) \le C\left(1 + \int_0^T\left(\|\partial_x\theta_\varepsilon/\theta_\varepsilon\|\,\|\theta_\varepsilon^{1/2}\|\right)^2\right)$$
$$\le C\left(1 + \int_0^T \|\partial_x\theta_\varepsilon/\theta_\varepsilon\|^2\right) \le C. \tag{3.5}$$

Hence,

$$\|\theta_\varepsilon\|_{L^1(0,T;L^\infty(\Omega))} \le C, \tag{3.6}$$

and finally, by interpolation with (3.2),

$$\|\theta_\varepsilon\|_{L^2(0,T;H)} \le C. \tag{3.7}$$

3.3 Third estimate

Taking (3.7) into account, it is now a standard matter to multiply (2.10) by $\partial_t\chi_\varepsilon$, integrate on Q_t, exploit the relation of $\beta_\varepsilon = \partial\Phi^\varepsilon$ and obtain the bound

$$\|\chi_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} + \sqrt{k}\|\chi_\varepsilon\|_{H^1(0,T;V)} \le C. \tag{3.8}$$

3.4 Fourth estimate

We multiply (2.9) by $\theta_\varepsilon + \partial_t\chi_\varepsilon$; we differentiate (2.10) with respect to t and multiply the result by $\partial_t\chi_\varepsilon$. We add the resulting equations and we integrate over Q_t; thanks to some cancellations, we obtain

$$\frac{1}{2}\|\theta_\varepsilon(t)\|^2 + \|\partial_x\theta_\varepsilon\|^2_{L^2(0,T;H)} + \frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|^2$$

$$+\|\partial_{xt}\chi_\varepsilon\|^2_{L^2(0,T;H)} + \frac{k}{2}\|\partial_{xt}\chi_\varepsilon(t)\|^2 + \int_0^t\int_\Omega \beta'_\varepsilon(\chi_\varepsilon)(\partial_t\chi_\varepsilon)^2$$

$$= \frac{1}{2}\|\theta_0\|^2 + \frac{1}{2}\|\partial_t\chi_\varepsilon(0)\|^2 + \frac{k}{2}\|\partial_{xt}\chi_\varepsilon(0)\|^2 + \sum_{i=5}^{9} I_i(t)\,, \tag{3.9}$$

where

$$I_5(t) := k\int_{Q_t}(\partial_{xt}\chi_\varepsilon)^2\theta_\varepsilon\,, \quad I_6(t) := -\int_{Q_t}(\theta_\varepsilon)^2\partial_t\chi_\varepsilon\,, \quad I_7(t) := \int_{Q_t}(\partial_t\chi_\varepsilon)^3\,,$$

$$I_8(t) := k\int_{Q_t}(\partial_{xt}\chi_\varepsilon)^2\partial_t\chi_\varepsilon\,, \qquad I_9(t) := -\int_{Q_t}\partial_x\theta_\varepsilon\partial_{xt}\chi_\varepsilon\,.$$

We set $t = 0$ in (2.10), obtaining

$$\partial_t\chi_\varepsilon(0) - k\partial_{xxt}\chi_\varepsilon(0) = \theta_0 - \theta_c + \partial_{xx}\chi_0 - \beta_\varepsilon(\chi_0). \tag{3.10}$$

We multiply (3.10) by $\partial_t\chi_\varepsilon(0)$ and integrate over Ω; we obtain that

$$\frac{1}{2}\|\partial_t\chi_\varepsilon(0)\|^2 + k\|\partial_{xt}\chi_\varepsilon(0)\|^2 \le \frac{1}{2}\|\theta_0 - \theta_c + \partial_{xx}\chi_0 - \beta_\varepsilon(\chi_0)\|^2. \tag{3.11}$$

Using also (1.10), (1.12), (1.13) and (1.14), we then get that the first three terms in the right-hand side of (3.9) are bounded by a constant independent of ε. As for the integrals terms, using Hölder inequality and (1.7), we have

$$|I_5(t)| \le k\int_0^t \|\partial_{xt}\chi_\varepsilon\|^2\|\theta_\varepsilon\|_{L^\infty(\Omega)}\,; \tag{3.12}$$

$$|I_6(t)| \le \int_0^t \|\theta_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_{L^\infty(\Omega)} \le c_2\int_0^t \|\theta_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_V$$

$$\le \frac{1}{8}\|\partial_t\chi_\varepsilon\|^2_{L^2(0,t;V)} + c\int_0^t \|\theta_\varepsilon\|^2\|\theta_\varepsilon\|^2\,; \tag{3.13}$$

$$|I_7(t)| \le \int_0^t \|\partial_t\chi_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_{L^\infty(\Omega)} \le c_2\int_0^t \|\partial_t\chi_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_V$$

$$\le \frac{1}{8}\|\partial_t\chi_\varepsilon\|^2_{L^2(0,t;V)} + c\int_0^t \|\partial_t\chi_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|^2\,; \tag{3.14}$$

$$|I_8(t)| \le k\int_0^t \|\partial_{xt}\chi_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_{L^\infty(\Omega)} \le kc_2\int_0^t \|\partial_{xt}\chi_\varepsilon\|^2\|\partial_t\chi_\varepsilon\|_V\,; \tag{3.15}$$

$$|I_9(t)| \le \frac{1}{2}\|\partial_{xt}\chi_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{2}\|\partial_x\theta_\varepsilon\|^2_{L^2(0,t;H)}\,. \tag{3.16}$$

Thanks to (3.8) and (3.6), we have that $k\|\partial_{xt}\chi_\varepsilon\|^2_{L^2(0,T;H)}$ is bounded independently of ε, $\|\theta_\varepsilon\|_{L^\infty(\Omega)} \in L^1(0,T)$, $\|\theta_\varepsilon\| \in L^2(0,T)$ and $\|\partial_t\chi_\varepsilon\| \in L^2(0,T)$. Hence, we can apply an extended version of Gronwall lemma to the function $\|\theta_\varepsilon(t)\|^2 + \|\partial_t\chi_\varepsilon(t)\|^2 + k\|\partial_{xt}\chi_\varepsilon(t)\|^2$ and obtain

$$\|\theta_\varepsilon\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leq C, \tag{3.17}$$

and

$$\|\chi_\varepsilon\|_{H^1(0,T;V)\cap W^{1,\infty}(0,T;H)} + \sqrt{k}\|\chi_\varepsilon\|_{W^{1,\infty}(0,T;V)} \leq C. \tag{3.18}$$

3.5 Fifth estimate

Multiplying (2.10) by $-\partial_{xx}\chi_\varepsilon$, integrating on Q_t and exploiting the monotonicity of β_ε, we obtain the bound

$$\|\chi_\varepsilon\|_{L^2(0,T;W)} + \sqrt{k}\|\chi_\varepsilon\|_{L^\infty(0,T;W)} \leq C. \tag{3.19}$$

3.6 Sixth estimate

By comparison in (2.10) and in (2.9), we also obtain

$$\|\beta_\varepsilon(\chi_\varepsilon)\|_{L^\infty(0,T;V')} \leq C\,, \tag{3.20}$$

$$\|\theta_\varepsilon\|_{H^1(0,T;V')} \leq C\,. \tag{3.21}$$

3.7 Passage to the limit

We sketch here the passage to the limit as $\varepsilon \downarrow 0$. Let us, however, detail a technical point. In the forthcoming limit procedure we will indeed use the following

$$\beta_\varepsilon \text{ converge to } \beta_{V,V'} \text{ in the sense of graphs in } V \times V'. \tag{3.22}$$

Namely, for all $u \in V$, $v \in V'$ such that $v \in \beta_{V,V'}(u)$ there exist $u_\varepsilon \in V$, $v_\varepsilon \in V'$, with $v_\varepsilon \in \beta_\varepsilon(u_\varepsilon)$, strongly converging to u, v, respectively, as ε goes to 0. The convergence (3.22) follows from (2.6) and the two easy facts:

$$\lim_{\varepsilon\downarrow 0} \Phi^\varepsilon_V(v) = \Phi_V(v)\,, \quad \forall v \in V, \tag{3.23}$$

$$\Phi_V(v) \leq \liminf_{\varepsilon\downarrow 0} \Phi^\varepsilon_V(v_\varepsilon)\,, \quad \forall v_\varepsilon \rightharpoonup v \text{ in } V, \tag{3.24}$$

and [1, Thm. 3.66, p. 373].

Taking into account well-known compactness results, the bounds (3.17)–(3.21) allow us to deduce the existence of a triplet of functions (θ, χ, η) such that (possibly passing to not relabeled subsequences) the following convergences hold

$$\theta_\varepsilon \rightharpoonup^* \theta \quad \text{in} \quad H^1(0,T;V') \cap L^2(0,T;V) \cap L^\infty(0,T;H), \tag{3.25}$$

$$\chi_\varepsilon \rightharpoonup^* \chi \quad \text{in} \quad W^{1,\infty}(0,T;V) \cap L^2(0,T;W), \tag{3.26}$$

$$\beta_\varepsilon(\chi_\varepsilon) \rightharpoonup \eta \quad \text{in} \quad L^2(0,T;V'). \tag{3.27}$$

Moreover, from (3.21), (3.17), (3.18), and the generalized Ascoli theorem (see, e.g., [18, Cor. 4]), we may also infer the strong convergences

$$\theta_\varepsilon \longrightarrow \theta \quad \text{in} \quad C^0([0,T];V') \cap L^2(0,T;H), \tag{3.28}$$

$$\chi_\varepsilon \longrightarrow \chi \quad \text{in} \quad C^0([0,T];H) \cap L^2(0,T;V). \tag{3.29}$$

Let us stress that, owing to (2.11), the convergence (3.28) entails that (1.24) holds.

Hence, we can pass to the limit in (2.10), and see that the properties (1.18)–(1.20) along with (1.22) are fulfilled by the triplet (θ, χ, η).

In order to prove (1.23), we multiply (2.10) by χ_ε in the duality pairing between V' and V and integrate from 0 to t. Thanks to (3.27) and (3.29), we find

$$\lim_{\varepsilon \downarrow 0} \int_0^t < \beta_\varepsilon(\chi_\varepsilon), \, \chi_\varepsilon >= \int_0^t < \eta, \chi > , \tag{3.30}$$

which, in view of [3, Prop. 1.1, p. 42] and of the graph convergence (3.22), implies (1.23).

Our next goal is to pass to the limit in (2.9). We remark that, from (3.25), (3.26) and (3.28), we achieve that

$$\theta_\varepsilon \, \partial_t \chi_\varepsilon \rightharpoonup^* \theta \, \partial_t \chi \quad \text{in} \quad L^\infty(0,T;H).$$

The critical terms are $(\partial_t \chi_\varepsilon)^2$ and $k(\partial_{xt} \chi_\varepsilon)^2$. In order to prove that

$$\lim_{\varepsilon \downarrow 0} \int_{Q_T} (\partial_t \chi_\varepsilon)^2 + k(\partial_{xt} \chi_\varepsilon)^2 = \int_{Q_T} (\partial_t \chi)^2 + k(\partial_{xt} \chi)^2 \tag{3.31}$$

(i.e., that $\partial_t \chi_\varepsilon$ actually converges strongly in $L^2(0,T;V)$ thanks to (3.26)), one has only to show that

$$\limsup_{\varepsilon \downarrow 0} \int_{Q_T} (\partial_t \chi_\varepsilon)^2 + k(\partial_{xt} \chi_\varepsilon)^2 \le \int_{Q_T} (\partial_t \chi)^2 + k(\partial_{xt} \chi)^2 \, . \tag{3.32}$$

The procedure is analogous to the one performed in [6, Sec. 4] and for the sake of brevity we omit the details. We only want to outline that the key point is to multiply both sides of (2.10) by $\partial_t \chi_\varepsilon$ and then exploit – besides the other convergences – (3.30). This completes the proof of Theorem 1.1.

3.8 Asymptotic analysis

We shall not give here a full proof of Thm. 1.2. Indeed the latter follows by observing that the above detailed estimates hold independently of both ε and k. Namely we are in a position to obtain (1.34)–(1.37) by standard compactness techniques and the passage to the limit in (2.9)–(2.10) can be eventually performed as above (the limits as $\varepsilon \downarrow 0$ and $k \downarrow 0$ being independent). As soon as a solution to (1.21)–(1.25) with $k = 0$ is obtained, the assertion of Thm. 1.2 follows from a comparison in (1.22) and standard parabolic estimates for the temperature. Analogously, further (formal, but easily justifiable) estimates in (1.30) yield the third of (1.27) as well as (1.28). In this connection, we also obtain the stronger inclusion (1.31), instead of (1.23), which holds as a consequence of (1.28) and [4, Prop. 2.5].

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Smoothing effect and strong L^2-wellposedness in the complex Ginzburg-Landau equation

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Abstract The complex Ginzburg-Landau equation is a complex-valued nonlinear heat equation. The exact form is obtained by normalizing the coefficient in front of the time derivative of the unknown function. The strong solvability in $L^2(\Omega)$, $\Omega \subset \mathbf{R}^N$, depends on the complex coefficient $\kappa + i\beta$ ($\kappa > 0$) of the nonlinear term $|u|^{q-2}u$ with $q \geq 2$. If $\kappa^{-1}|\beta| \leq 2\sqrt{q-1}/(q-2)$, then monotonicity methods are available, without any upper bound on q. On the other hand, if $\kappa^{-1}|\beta| > 2\sqrt{q-1}/(q-2)$, then either compactness or contraction methods are required, as its consequence q has to be bounded above: $q \leq 2 + 4/N$.

1 Introduction

The complex Ginzburg-Landau equation (CGLeq) is a semilinear parabolic partial differential equation. In this lecture we are concerned with the *strong L^2-wellposedness* of initial-boundary value problems for (CGLeq):

$$\text{(CGL)} \qquad \begin{cases} \partial_t u - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-2}u - \gamma u = 0 \ \text{ on } \ \Omega \times \mathbf{R}_+, \\ u = 0 \ \text{ on } \ \partial\Omega \times \mathbf{R}_+, \\ u(x, 0) = u_0(x), \ x \in \Omega. \end{cases}$$

Here $\mathbf{R}_+ := (0, \infty)$ and $\Omega \subset \mathbf{R}^N$ is a (bounded or unbounded) domain with boundary $\partial\Omega$. Since $i := \sqrt{-1}$, the unknown u is a complex-valued function with respect to $(x, t) \in \Omega \times [0, \infty)$, with $\partial_t u := \partial u/\partial t$. (CGLeq) is introduced to describe dissipative physical systems (cf. Aranson-Kramer [4]) while the (real) Ginzburg-Landau equation is already well known in connection with the superconductivity (cf. Mielke [24]). That is, the adjective "complex" symbolizes the difference between two kinds of Ginzburg-Landau equations

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though the term "complex" is absent in the pioneering works on the L^2-wellposedness of problem (CGL) (see, e.g., [34, Section IV-5], [36]). (CGLeq) may be regarded as a *parabolic evolution equation* in $L^2(\Omega) = L^2(\Omega, \mathbf{C})$ since we assume as usual that $\lambda, \kappa \in \mathbf{R}_+$. As for other parameters we assume that $\alpha, \beta, \gamma \in \mathbf{R}$ and $q \in [2, \infty)$. By the parabolicity we may expect the *strong* solvability of (CGL) and its *smoothing effect* on L^2-initial data and more.

On the one hand, (CGLeq) is known as an *amplitude equation* derived by reduction from the fundamental systems of equations describing several physical phenomena. On the other hand, one glance at the form tells us that (CGLeq) is simply a nonlinear *complex* heat equation and contains as a particular case ($\lambda = \kappa = 1$, $\alpha = \beta = \gamma = 0$) the nonlinear (real) heat equation

$$\text{(NLHeq)}_+ \qquad \partial_t u - \Delta u + |u|^{q-2}u = 0 \ \text{ on } \ \Omega \times \mathbf{R}_+.$$

Equation (NLHeq)_+ with $|u|^{q-2}u$ replaced with $-|u|^{q-2}u$ (that is, (CGLeq) with "$\kappa = -1$") has been holding the attention of research workers to study blow-up of local solutions of this new equation (NLHeq)_-, while in (CGL) the uniqueness of solutions for large $|\beta|$ (even if $\kappa > 0$) has not yet been solved satisfactorily. This may be explained from the fact that (CGLeq) is actually a strongly coupled system in the real product space $L^2(\Omega, \mathbf{R}) \times L^2(\Omega, \mathbf{R})$:

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} - \begin{pmatrix} \lambda & -\alpha \\ \alpha & \lambda \end{pmatrix}\Delta\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} \kappa & -\beta \\ \beta & \kappa \end{pmatrix}\left|v^2 + w^2\right|^{(q-2)/2}\begin{pmatrix} v \\ w \end{pmatrix} - \gamma\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In particular, the case with large $|\beta|$ is most complicated.

The investigation for the existence of (weak) solutions to (CGL) began with Temam [34], Yang [36] at the end of 1980s. In the middle of 1990s systematic studies were done by Levermore-Oliver [20], [21] and Ginibre-Velo [12], [13]. The purpose of this lecture is to report the main results in [29] and [30]. In [29] the smoothing effect on L^2-initial data (which had been folklore for a long time) was first established for (CGL) even though a strong restriction is imposed on the complex coefficient $\kappa + i\beta$ in front of the nonlinear term (see Theorem 1.1 below). In the latest [30] the strong L^2-wellposedness (including smoothing effect) has been proved for (CGL) with bounded Ω and arbitrary coefficient $\kappa + i\beta$ (see Theorem 1.2). Here the L^2-wellposedness contains *continuous dependence* of solutions on L^2-initial data. In this connection it should be noted that if $N = 2$, then the uniqueness of (weak) solutions to (CGL) has already been shown by Ogawa-Yokota [25] (see also [13] and Machihara-Nakamura [23] for the case of $\Omega = \mathbf{R}^N$).

Since we have assumed that $\lambda, \kappa \in \mathbf{R}_+$, the nonlinear Schrödinger equation

$$\text{(NLSeq)} \qquad \partial_t u - i\Delta u + i|u|^{q-2}u = 0 \ \text{ on } \ \Omega \times \mathbf{R}_+$$

is not a particular case of (CGLeq) with $\alpha = \beta = 1$. Nevertheless, the following "singular perturbation (or inviscid limit) problem" seems to be important:

$$\text{(NLS)} = \lim_{\substack{\lambda \downarrow 0 \\ \kappa \downarrow 0}} \text{(CGL)}? \tag{1.1}$$

Here (NLS) means the initial-boundary value problem for (NLSeq) in the same way as in the case of (CGL). Concerning the convergence (1.1) the investigation is in rapid progress especially in the case of $\Omega = \mathbf{R}^N$ (cf. [6], [23], [25] and [35]). However, we shall not consider the problem (1.1) and the detailed properties of solutions to (CGL) such as their large time behavior (cf. [15]), their regularity (cf. [17]) and the existence of global attractors (cf. [10] and [34]).

In order to state the results we need the following

DEFINITION 1.1 *Let $Y = L^2(\Omega)$ or $H^s(\Omega)$ $(s < 0)$. Then $u \in C([0,\infty); Y)$ is a strong solution to (CGL) with initial value $u_0 \in Y$ if u has the following four properties:*

(a) $u(t) \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{2(q-1)}(\Omega)$ a.a. $t > 0$;

(b) $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega))$ ($\Longrightarrow$ u is strongly differentiable a.e. on $\mathbf{R}_+$);

(c) u satisfies (CGLeq) in $L^2_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega))$;

(d) $u(0) = u_0$.

According to the local existence theory developed in [21] (CGL) with $u_0 \in H^s(\Omega)$ $(s < 0)$ on $[0,\infty)$ may be regarded as a new (CGL) with initial value $u(\varepsilon) \in L^2(\Omega)$ on $[\varepsilon,\infty)$ for any $\varepsilon > 0$.

In what follows we assume for simplicity that $\partial\Omega$ is bounded and of class C^2. In the first theorem monotonicity methods apply to give a strong result in the sense that

(i) there is no restriction on the power $q \geq 2$ of the nonlinearity;

(ii) there is an extension to the quasi-linear case if Ω is bounded.

THEOREM 1.1 ([29, Theorem 1.3 with $p = 2$]). *Let $q \geq 2$. For the coefficient $\kappa + i\beta$ assume that*

$$\kappa^{-1}|\beta| \in I_q := \begin{cases} [0,\infty) & (q = 2), \\ \left[0, \dfrac{2\sqrt{q-1}}{q-2}\right] & (2 < q < \infty). \end{cases} \tag{1.2}$$

Then for any initial value $u_0 \in L^2(\Omega)$ *there exists a unique strong solution* $u(t) = u(x,t)$ *to (CGL) such that* $u \in C([0,\infty); L^2(\Omega))$,

$$\begin{aligned} &u \in C^{0,1/2}_{\mathrm{loc}}(\mathbf{R}_+; H^1_0(\Omega)) \cap C^{0,1/q}_{\mathrm{loc}}(\mathbf{R}_+; L^q(\Omega)), \\ &\partial_t u,\ \Delta u,\ |u|^{q-2}u \in L^\infty_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega)), \\ &u(t) \in H^2(\Omega) \cap H^1_0(\Omega) \cap L^{2(q-1)}(\Omega), \quad \forall\, t > 0. \end{aligned}$$

The family $\{U(t)\}_{t\ge 0}$ *of solution operators* $U(t)$, *defined by*

$$(U(t)u_0)(x) := u(t,x), \quad \forall\, t \ge 0, \ \text{a.a. } x \in \Omega,$$

forms a semigroup of quasi-contractions on $L^2(\Omega)$*:*

$$\|U(t)u_0 - U(t)v_0\|_{L^2} \le e^{\gamma t}\|u_0 - v_0\|_{L^2}, \quad t \ge 0,\ u_0, v_0 \in L^2(\Omega). \tag{1.3}$$

In the second theorem there is no restriction on the complex coefficient $\kappa + i\beta$, while a strong restriction is imposed on the power q. Thus, the second theorem makes sense when $\kappa^{-1}|\beta| \notin I_q$ ($q \ge 2$). The proof may be regarded as a combination of compactness methods and local Lipschitz continuity of the nonlinear term.

THEOREM 1.2 ([30, Theorem 1.1]). *Assume that* Ω *is* **bounded**, *and*

$$2 \le q \le 2 + \frac{4}{N}. \tag{1.4}$$

Then for any initial value $u_0 \in L^2(\Omega)$ *there exists a unique strong solution* $u(t) = u(x,t)$ *to (CGL), with norm bound*

$$\|u(t)\|_{L^2} \le e^{\gamma t}\|u_0\|_{L^2} \quad \forall\, t \ge 0, \tag{1.5}$$

such that $u \in C([0,\infty); L^2(\Omega))$,

$$\begin{aligned} &u \in C^{0,1/2}_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega)) \cap C(\mathbf{R}_+; H^1_0(\Omega)), \\ &\partial_t u, \Delta u, |u|^{q-2}u \in L^2_{\mathrm{loc}}(\mathbf{R}_+; L^2(\Omega)). \end{aligned}$$

Moreover, the family $\{U(t)\}_{t\ge 0}$ *of solution operators forms a semigroup of locally Lipschitz continuous operators on* $L^2(\Omega)$: *for* $u_0, v_0 \in B_M$,

$$\|U(t)u_0 - U(t)v_0\|_{L^2} \le L_{M,T}e^{K_1 t}\|u_0 - v_0\|_{L^2}, \quad 0 \le t \le T. \tag{1.6}$$

Here $\log(L_{M,T}) := K_2 M^2 e^{2\gamma_+ T}$ $(\gamma_+ := \gamma \vee 0)$ *and*

$$B_M := \{u \in L^2(\Omega);\ \|u\|_{L^2} \le M\} \quad (M > 0).$$

The constants K_j, $j = 1, 2$, depend only on $\lambda, \kappa, \beta, \gamma, q, N$.

REMARK 1.1 Theorems 1.1 and 1.2 are respectively the consequences of two kinds of abstract theorems ([29, Theorem 5.2] and [30, Theorem 2.2]). These abstract theorems will be unified in a forthcoming paper [31] (see the concluding remark in Section 5). In the unified theorem for (CGL) we shall eliminate the boundedness of Ω assumed in Theorem 1.2.

REMARK 1.2 It follows from (1.5) and (1.6) with $\gamma \leq 0$ that $\{U(t)\}$ is a concrete example of Lipschitz semigroup on B_M in the sense of Kobayashi-Tanaka [19]: $U(t)$ leaves B_M invariant, with

$$\|U(t)u_0 - U(t)v_0\|_{L^2} \leq e^{K_2 M^2} e^{K_1 t} \|u_0 - v_0\|_{L^2} \quad \forall\, t \geq 0, \quad \forall\, u_0, v_0 \in B_M.$$

In this connection the resolvent problem is discussed in [27].

Next, let $N = 1$ and $\Omega = I := (a, b)$ in (CGL). Then, combining Theorems 1.1 and 1.2 with the idea of Levermore-Oliver [21] to construct local solutions to (CGL) with distribution-valued initial data (i.e., $u_0 \in H^s(I)$, $s < 0$), we obtain

COROLLARY 1.1 ([30]). *Let $N = 1$, $\Omega = I$. For the exponent "$s \in [-2, 0)$" of $H^s(I)$ assume that the power "$q \in [2, 6)$" satisfies the following constraint*:

$$2 \leq q < g(s) := \begin{cases} 1 - \dfrac{2}{s} & \text{if } -2 \leq s \leq -1, \\ 1 + \dfrac{6}{1-2s} & \text{if } -1 \leq s \leq -\dfrac{1}{2}, \\ 2 + \dfrac{4}{1-2s} & \text{if } -\dfrac{1}{2} \leq s < 0. \end{cases} \tag{1.7}$$

Then for any $u_0 \in H^s(I)$ there exists a unique strong solution $u(t) = u(x,t)$ to (CGL) such that $u \in C([0, \infty); H^s(I))$,

$$u \in C^{0,1/2}_{\mathrm{loc}}(\mathbf{R}_+; L^2(I)) \cap C(\mathbf{R}_+; H^1_0(I)),$$

$$\partial_t u, \Delta u, |u|^{q-2}u \in L^2_{\mathrm{loc}}(\mathbf{R}_+; L^2(I)).$$

EXAMPLE 1.1 Let $0 \in I$ and δ be the Dirac measure. Then $\delta \in H^s(I)$ if $s < -1/2$. Therefore (1.7) guarantees that we can choose $u_0 = \delta$ if $q \in [2, 4)$ (for the real space case see Brézis-Friedman [9]).

Let $g \in C[-2,0)$ be as in (1.7). Then $g(s)$ is an upper bound for admissible powers of the nonlinear term. Therefore, if $N = 1$ then g is extended to a continuous function on $[-2,0]$ if we define $g(0) := 6$. In fact, if $s = 0$, that is, $u_0 \in L^2(I)$, then the upper bound is $2 + 4/N = 6$ for $N = 1$ (see (1.4)).

After the completion of [30] Professor Herbert Amann informed the author of his contribution to (CGL) in $L^p(\Omega)$ (private communication):

PROPOSITION 1.1 *Assume that Ω is* **bounded**, *$\kappa \in \mathbf{R}$ and*

$$2 \le q \le 2 + \frac{2p}{N}, \quad 1 \le p < \infty. \tag{1.8}$$

Then for any initial value $u_0 \in L^p(\Omega)$ there exists a unique local C^1-solution $u(t) = u(x,t)$ to (CGL), that is, there exists $T = T_{\max} > 0$ such that

$$u \in C([0,T); L^p(\Omega)) \cap C^1((0,T); L^p(\Omega)).$$

The proof is based on his local theory for semilinear parabolic problems developed in [1]–[3].

REMARK 1.3 As a consequence of Proposition 1.1 with $p = 2$, we can assert in Theorem 1.2 that $u \in C([0,\infty); L^2(\Omega)) \cap C^1(\mathbf{R}_+; L^2(\Omega))$.

---

2 Preliminaries

Our strategy for the proofs of Theorems 1.1 and 1.2 is to rewrite (CGLeq) as an abstract evolution equation in a *complex* Hilbert space X. Defining the *m-accretive* (maximal monotone) *operators* S, B in $X := L^2(\Omega) = L^2(\Omega, \mathbf{C})$ as (for S see Brézis [8, Théorème IX.25]):

$$Su := -\Delta u, \quad D(S) := H_0^1(\Omega) \cap H^2(\Omega), \tag{2.1}$$

$$Bu := |u|^{q-2}u, \quad D(B) := L^{2(q-1)}(\Omega) \cap L^2(\Omega), \tag{2.2}$$

(CGL) is formulated in the form of initial value problems for ordinary differential equations with operator-coefficients, which will simply be called the **abstract Cauchy problem**:

$$(\mathbf{ACP}) \begin{cases} D_t u + (\lambda + i\alpha)Su(t) + (\kappa + i\beta)Bu(t) - \gamma u(t) = 0, \text{ a.e. on } \mathbf{R}_+, \\ u(0) = u_0. \end{cases}$$

Here $D_t u := du/dt$ and we call $u \in C([0,\infty); X)$ a *strong solution* to (**ACP**) with $u_0 \in X$ if u has the following four properties:

(a) $u(t) \in D(S) \cap D(B)$ a.a. $t > 0$;

(b) $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}_+; X)$ ($\Longrightarrow u$ is strongly differentiable a.e. on $\mathbf{R}_+$);

(c) u satisfies (ACPeq) in $L^2_{\mathrm{loc}}(\mathbf{R}_+; X)$;

(d) $u(0) = u_0$.

In the proof of Theorem 1.1 we shall apply the celebrated Kōmura-Kato generation theorem for semigroups of nonlinear contractions on Hilbert spaces (this theorem is generalized to the Crandall-Liggett theorem). In other words, the solvability of (**ACP**) is reduced to the **abstract stationary problem**:

$$(\mathbf{AStP}) \qquad u + (\lambda + i\alpha)Su + (\kappa + i\beta)Bu = v.$$

To show that $(\lambda + i\alpha)S + (\kappa + i\beta)B$ is m-accretive in X, we may adopt the **approximate stationary problem**:

$$(\mathbf{AStP})_\varepsilon \qquad u_\varepsilon + (\lambda + i\alpha)S_\varepsilon u_\epsilon + (\kappa + i\beta)Bu_\varepsilon = v,$$

where S_ε is the *Yosida approximation* of S: $S_\varepsilon := \varepsilon^{-1}(1 - (1 + \varepsilon S)^{-1})$. It is worth noticing that $(\lambda + i\alpha)S$ is m-accretive in X for every $\lambda \geq 0$ and $\alpha \in \mathbf{R}$ because S is actually nonnegative selfadjoint. Since $(\lambda + i\alpha)S_\varepsilon$ is bounded and accretive, $(\mathbf{AStP})_\varepsilon$ is uniquely solvable if the m-accretivity of B is preserved under the multiplication by $\kappa + i\beta$. (The operator $(\lambda + i\alpha)S + (\kappa + i\beta)B_\varepsilon$ also works similarly.)

In the proof of Theorem 1.2 we start with the approximate evolution problem $(\mathbf{ACP})_\varepsilon$, which is defined as (**ACP**) with B replaced with its Yosida approximation B_ε. Since B_ε is Lipschitz continuous on X, $(\mathbf{ACP})_\varepsilon$ is always uniquely solvable in X.

Next, we have to pay attention to the aspect of S, B and B_ε as *subdifferential* operators in X. Namely, it is well known that $S = \partial\varphi$ and $B = \partial\psi$, where φ and ψ are proper lower semi-continuous convex functions on X:

$$\varphi(u) := \begin{cases} (1/2)\|\nabla u\|^2_{L^2} & \text{if } u \in H^1_0(\Omega), \\ +\infty & \text{otherwise}, \end{cases}$$

$$\psi(u) := \begin{cases} (1/q)\|u\|^q_{L^q} & \text{if } u \in L^q(\Omega) \cap L^2(\Omega), \\ +\infty & \text{otherwise}. \end{cases}$$

In order to understand B_ε it is indispensable to start with *Moreau-Yosida regularization* of ψ:

$$\psi_\varepsilon(v) := \inf\left\{\psi(w) + \frac{1}{2\varepsilon}\|w - v\|^2; w \in L^2(\Omega)\right\}, \quad \varepsilon > 0.$$

In fact, it can be shown that

$$\psi_\varepsilon(v) = \psi((1+\varepsilon B)^{-1}v) + (\varepsilon/2)\|B_\varepsilon v\|^2 \le \psi(v), \quad \forall\, v \in L^2(\Omega),$$

ψ_ε is a convex function of class C^1 with Fréchet derivative $\partial(\psi_\varepsilon)$, which is equal to B_ε:

$$\partial\psi_\varepsilon := \partial(\psi_\varepsilon) = \psi_\varepsilon' = B_\varepsilon = (\partial\psi)_\varepsilon \tag{2.3}$$

(cf. [5], [7] and [32]). This viewpoint is essential when we try to prove smoothing effect of the solution operators, that is, $U(t)u_0 = u(t) \in D(S) \cap D(B)$ (a.a. $t > 0$) even if $u_0 \in X$. Thus, in the proof of Theorem 1.2 we adopt the **approximate Cauchy problem**:

$$(\mathbf{ACP})_\varepsilon \quad \begin{cases} D_t u_\varepsilon + (\lambda + i\alpha)\partial\varphi(u_\varepsilon(t)) + (\kappa + i\beta)\partial\psi_\varepsilon(u_\varepsilon(t)) - \gamma u_\varepsilon(t) = 0 \\ \hfill \text{a.e. on } \mathbf{R}_+, \\ u_\varepsilon(0) = u_0. \end{cases}$$

---

3 Proof of Theorem 1.1

In this section we first give a sufficient condition guaranteeing the m-accretivity of the linear combination of two m-accretive operators (S and B in X as in Section 2) with complex coefficients and then take into account the character of S and B as subdifferential operators. Namely, we begin by showing that $A + \gamma$ is m-accretive in X when we define

$$A := (\lambda + i\alpha)S + (\kappa + i\beta)B - \gamma, \quad D(A) := D(S) \cap D(B). \tag{3.1}$$

We have to answer to the following questions:

(i) the accretivity of $(\kappa + i\beta)B$;

(ii) the maximality of $A + \gamma$.

Now let B be a nonlinear operator in a complex Hilbert space X. Then B is said to be *sectorially-valued* in the sense of Kato [18, Section V.3.10] if

$$|\mathrm{Im}\,(Bu_1 - Bu_2, u_1 - u_2)_X| \le (\tan\omega_q)\mathrm{Re}\,(Bu_1 - Bu_2, u_1 - u_2)_X \quad \forall\, u_1, u_2 \in D(B). \tag{3.2}$$

(We assume for simplicity that B is single-valued, with $0 < \omega_q < \pi/2$.) It is easy to see that (3.2) is equivalent to the following condition:

$$\mathrm{Re}\,(e^{i\theta}(Bu_1 - Bu_2), u_1 - u_2)_X \ge 0 \;\; \forall\, \theta \text{ with } |\theta| \le \omega_q' := \pi/2 - \omega_q. \tag{3.3}$$

Here we can state two abstract theorems which lead us to Theorem 1.1. The first theorem is concerned with the solvability of $(\mathbf{ACP})$ with $u_0 \in D(A)$.

THEOREM 3.1 ([28]). *Let S be a nonnegative selfadjoint operator in X, B an m-accretive operator in X, satisfying* (3.2). *Assume that $D(S) \cap D(B) \neq \emptyset$. Assume further that*

$$|\mathrm{Im}\,(S_\varepsilon u, Bu)_X| \leq (\tan\omega_q)\mathrm{Re}\,(S_\varepsilon u, Bu)_X, \quad \forall\, u \in D(B), \ \forall\, \varepsilon > 0. \quad (3.4)$$

If $\kappa^{-1}|\beta| \leq 1/\tan\omega_q$, then $(\lambda+i\alpha)S+(\kappa+i\beta)B$ is m-accretive in X, so that $-A$ defined as (3.1) generates a quasi-contraction semigroup $\{U(t)\}$ (of type γ) on $\overline{D(A)}$:

$$\|U(t)u - U(t)v\|_X \leq e^{\gamma t}\|u-v\|_X, \quad \forall\, t \geq 0, \ \forall\, u, v \in \overline{D(A)}, \quad (3.5)$$

*where $\overline{D(A)}$ is the closure of $D(A)$. In particular, if $u_0 \in D(A)$, then $u(t) := U(t)u_0$ is the unique strong solution to (**ACP**), with property $U(t)u_0 \in D(A)$ for every $t \geq 0$.*

The second theorem is concerned with (**ACP**) with $u_0 \in X$.

THEOREM 3.2 ([29]). *Under the setting of Theorem 3.1 assume that B is given by the form of subdifferential: $B = \partial\psi$ (and regard $S = \partial\varphi$), in which ψ satisfies homogeneity of degree q :*

$$\exists\, q \in [2,\infty); \ \psi(\zeta u) = |\zeta|^q\psi(u) \ \forall\, u \in D(\psi), \quad \forall\, \zeta \in \mathbb{C} \ \textit{with} \ \mathrm{Re}\,\zeta > 0.$$

Then the semigroup $\{U(t)\}$ has **smoothing effect** *on the initial data:*

$$U(t)\overline{D(A)} \subset D(A), \quad \forall\, t > 0,$$

*and $u(t) := U(t)u_0$ for $u_0 \in \overline{D(A)}$ is the unique strong solution to (**ACP**).*

REMARK 3.1 In [29] we have actually replaced (3.4) with

$$|\mathrm{Im}\,(Su, \partial\psi_\varepsilon(u))_X| \leq (\tan\omega_q)\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_X, \ \forall\, u \in D(S), \ \forall\, \varepsilon > 0. \quad (3.6)$$

Under condition (3.2) it is not so difficult to show that (3.4) is equivalent to (3.6) (for a proof see [31]).

OUTLINE OF PROOFS It follows from the sectorial-valuedness of B that if $\kappa^{-1}|\beta| \leq 1/\tan\omega_q$ ($\kappa > 0$), then $(\kappa+i\beta)B$ is accretive in X:

$$\begin{aligned} &\mathrm{Re}((\kappa+i\beta)(Bu_1 - Bu_2), u_1 - u_2)_X \\ &= \kappa\,\mathrm{Re}(Bu_1 - Bu_2, u_1 - u_2)_X - \beta\,\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)_X \\ &\geq \left(\frac{\kappa}{\tan\omega_q} - |\beta|\right)|\mathrm{Im}(Bu_1 - Bu_2, u_1 - u_2)_X| \geq 0. \end{aligned}$$

As for the maximality of $(\kappa + i\beta)B$ (i.e., $R(1 + (\kappa + i\beta)B) = X$, which is equivalent to that of B) it suffices to note that the m-accretivity of B is reduced to that of the mapping $z \mapsto |z|^{q-2}z$ on $\mathbf{C}$ (cf. [28, Lemma 3.1]). Thus, as mentioned in Section 2, for every $v \in X$ there is a unique solution $u_\varepsilon \in D(B)$ to $(\mathbf{AStP})_\varepsilon$. Next, it follows from (3.4) that

$$\lambda\|S_\varepsilon u\|_X \leq \|(\lambda + i\alpha)S_\varepsilon u + (\kappa + i\beta)Bu\|_X, \quad \forall\, u \in D(B),\ \lambda > 0,$$

while the boundedness of $\{u_\varepsilon\}$ follows from the condition $D(S) \cap D(B) \neq \emptyset$. In this way $(\mathbf{AStP})_\varepsilon$ yields the boundedness of $\{S_\varepsilon u_\varepsilon\}$: $\lambda\|S_\varepsilon u_\varepsilon\|_X \leq \|v\|_X + \|u_\varepsilon\|_X$, so that we can show that $\{u_\varepsilon\}$ satisfies Cauchy condition for convergence. Since both of $\{S_\varepsilon u_\varepsilon\}$ and $\{Bu_\varepsilon\}$ converge weakly as $\varepsilon \downarrow 0$, $u := \lim_{\varepsilon\downarrow 0} u_\varepsilon$ is proved to be a unique solution to $(\mathbf{AStP})$ (the weak closedness of S and the demi-closedness of B in X). This concludes the m-accretivity of $A + \gamma = (\lambda + i\alpha)S + (\kappa + i\beta)B$ and the generation of the corresponding semigroup $\{U(t)\}$ on $\overline{D(A)}$.

However, Theorem 3.1 guarantees the existence of unique strong solutions to $(\mathbf{ACP})$ just for initial values from $D(A) = D(S) \cap D(B)$ (usual theory of nonlinear (quasi-)contraction semigroups).

Now, under the setting of Theorem 3.2, we consider the sequence $\{u_n(\cdot)\} = \{U(\cdot)u_{0,n}\}$ of solutions to $(\mathbf{ACP})$ with initial value $\{u_{0,n}\} \subset D(A)$, where $\{u_{0,n}\}$ is an approximate sequence for $u_0 \in \overline{D(A)}$:

$$(\mathbf{ACP})_n \quad \begin{cases} D_t u_n + (\lambda + i\alpha)(Su_n)(t) + (\kappa + i\beta)\partial\psi(u_n(t)) - \gamma u_n(t) = 0 \\ \hfill \text{a.e. on } \mathbf{R}_+, \\ u_n(0) = u_{0,n}. \end{cases}$$

We see from (3.5) that $U(\cdot)u_0 = \lim_{n\to\infty} u_n(\cdot)$ in $C([0,T];X)$ for every $T > 0$. The proof of the smoothing effect $U(t)u_0 = \lim_{n\to\infty} u_n(t) \in D(A)$ $(t > 0)$ is based on the equality:

$$D_t(\psi \circ u_n)(t) = \mathrm{Re}\big(\partial\psi(u_n(t)), D_t u_n(t)\big)_X, \quad \text{for a.e. on } (0,\infty).$$

Setting $\eta := (\lambda^2 + \alpha^2)^{-1}\lambda\kappa$, we can show that $\{Su_n\}$, $\{\partial\psi(u_n)\}$ are bounded in $L^2(T^{-1}, T; X)$ for all $T > 1$:

$$\int_{T^{-1}}^{T} (\|(Su_n)(t)\|_X^2 + \eta\|\partial\psi(u_n(t))\|_X^2)\, dt \leq MT(1 + T^2)e^{kT}\|u_{0,n}\|_X^2,$$

where $M > 0$ and $k > 2\gamma_+ := 2\max\{\gamma, 0\}$ are two constants. In view of $(\mathrm{ACPeq})_n$, $\{D_t u_n\}$ is also bounded in $L^2(T^{-1}, T; X)$. Therefore $U(t)u_0$, the limit of $u_n(t)$, satisfies (ACPeq) a.e. on (T^{-1}, T). In fact, D_t and S are weakly closed, while $\partial\psi$ is demi-closed in $L^2(T^{-1}, T; X)$. Since $T > 1$ is arbitrary, $U(t)u_0$ is a strong solution to $(\mathbf{ACP})$. But, what we have proved is

$$U(t)u_0 \in D(A) \text{ for a.e. on } (0,\infty),\ \forall\, u_0 \in \overline{D(A)}. \tag{3.7}$$

Nevertheless, we can remove the term "a.e." in (3.7). In fact, we can choose $t_0 > 0$ in "$U(t_0)u_0 \in D(A)$" as small as possible so that we can take $U(t_0)u_0$ as a new initial value in Theorem 3.1 to conclude that $U(t)u_0 \in D(A)$ $(t \geq t_0)$.

In what follows we shall verify the inequalities (3.4) and (3.2) when S, B are given by (2.1), (2.2), respectively.

Verification of (3.4). Let z, $y \in \mathbf{C}$ with $z \neq y$. Then in [22, Lemma 2.2] Liskevich-Perelmuter derived the following remarkable inequality (a simpler proof is given in [29, Lemma 2.1]):

$$\frac{|\mathrm{Im}\,(\overline{z}-\overline{y})(|z|^{q-2}z-|y|^{q-2}y)|}{\mathrm{Re}\,(\overline{z}-\overline{y})(|z|^{q-2}z-|y|^{q-2}y)} \leq \frac{|q-2|}{2\sqrt{q-1}}, \quad 1<q<+\infty.$$

Setting $z := v(x)$, $y := w(x)$ and integrating it on Ω, we have a new inequality for complex-valued functions: for $v, w \in L^q(\Omega)$ $(v \neq w)$

$$\frac{|\mathrm{Im}\,\langle v-w, |v|^{q-2}v-|w|^{q-2}w\rangle_{L^q,L^{q'}}|}{\mathrm{Re}\,\langle v-w, |v|^{q-2}v-|w|^{q-2}w\rangle_{L^q,L^{q'}}} \leq \frac{|q-2|}{2\sqrt{q-1}}, \quad \frac{1}{q}+\frac{1}{q'}=1; \tag{3.8}$$

note that $|v|^{q-2}v \in L^{q'}(\Omega)$ $\forall$ $v \in L^q(\Omega)$. Next, we utilize the m-accretive realization A_q of the minus Laplacian (with Dirichlet condition) in $L^q(\Omega)$:

$$A_q := -\Delta \ \text{ with } \ D(A_q) := W^{2,q}(\Omega)\cap W_0^{1,q}(\Omega), \quad 1<q<\infty.$$

Then we can show that A_q is sectorial of type $S(\tan\omega)$,

$$\omega := \tan^{-1}\Big(\frac{|q-2|}{2\sqrt{q-1}}\Big),$$

in the sense of Goldstein [14, Definition 1.5.8, p. 37] (cf. Henry [16, p. 32]; see also [26]): for all $u \in D(A_q)$,

$$|\mathrm{Im}\langle A_q u, |u|^{q-2}u\rangle_{L^q,L^{q'}}| \leq \frac{|q-2|}{2\sqrt{q-1}}\mathrm{Re}\langle A_q u, |u|^{q-2}u\rangle_{L^q,L^{q'}}. \tag{3.9}$$

It seems to be a surprise that the constant in (3.8) and (3.9) coincides. Such a coincidence will be explained as a proof of (3.4) in the rest of this section.

Now we want to prove (3.9) with A_q replaced with its Yosida approximation $A_{q,\varepsilon}$: for all $v \in L^q(\Omega)$,

$$|\mathrm{Im}\langle A_{q,\varepsilon}v, |v|^{q-2}v\rangle_{L^q,L^{q'}}| \leq \frac{|q-2|}{2\sqrt{q-1}}\mathrm{Re}\langle A_{q,\varepsilon}v, |v|^{q-2}v\rangle_{L^q,L^{q'}}. \tag{3.10}$$

The proof of (3.10) is simply performed by combining (3.8) with (3.9). Noting that $(1+\varepsilon A_q)^{-1}v \in L^q(\Omega)$ for every $v \in L^q(\Omega)$, we can express the pairing

$\langle A_{q,\varepsilon}v, |v|^{q-2}v\rangle_{L^q,L^{q'}}$ as follows:

$$\begin{aligned}&\langle A_{q,\varepsilon}v, |v|^{q-2}v\rangle_{L^q,L^{q'}}\\ &= \langle A_q(1+\varepsilon A_q)^{-1}v,\ |(1+\varepsilon A_q)^{-1}v|^{q-2}(1+\varepsilon A_q)^{-1}v\rangle_{L^q,L^{q'}}\\ &\quad + \varepsilon^{-1}\langle v-(1+\varepsilon A_q)^{-1}v,\ |v|^{q-2}v - |(1+\varepsilon A_q)^{-1}v|^{q-2}(1+\varepsilon A_q)^{-1}v\rangle_{L^q,L^{q'}},\end{aligned}$$

where we have used the definition of Yosida approximation. Here we interpret one and the same element $(1+\varepsilon A_q)^{-1}v$ on the right-hand side in two different ways, namely as $u := (1+\varepsilon A_q)^{-1}v \in D(A_q)$ in the first term and as $w := (1+\varepsilon A_q)^{-1}v \in L^q(\Omega)$ in the second term. In this way we can write down the pairing as follows:

$$\begin{aligned}&\langle A_{q,\varepsilon}v, |v|^{q-2}v\rangle_{L^q,L^{q'}}\\ &= \langle A_q u, |u|^{q-2}u\rangle_{L^q,L^{q'}} + \varepsilon^{-1}\langle v-w, |v|^{q-2}v - |w|^{q-2}w\rangle_{L^q,L^{q'}}.\end{aligned}$$

Applying (3.9) and (3.8) to the first and second terms on the right-hand side, respectively, we can obtain (3.10).

Finally, let $u \in D(B)$, $q \geq 2$. Then we have

$$(S_\varepsilon u, Bu)_{L^2} = \langle A_{q,\varepsilon}u, |u|^{q-2}u\rangle_{L^q,L^{q'}}, \quad \forall\, u \in D(B). \tag{3.11}$$

In fact, noting that $D(B) = L^2(\Omega)\cap L^{2(q-1)}(\Omega) \subset L^q(\Omega)$, we can conclude that $S_\varepsilon u = A_{q,\varepsilon}u \in L^2(\Omega)\cap L^q(\Omega)$ and $Bu = |u|^{q-2}u \in L^2(\Omega)\cap L^{q'}(\Omega)$. This proves (3.11). By virtue of (3.10) we can obtain (3.4) with $X = L^2(\Omega)$ and

$$\tan\omega_q = c_q := \frac{q-2}{2\sqrt{q-1}};$$

in this connection see also Remark 4.2 below.

Verification of (3.2). (3.2) is also derived from (3.8) in the same way as above.

4 Proof of Theorem 1.2

Given a nonnegative selfadjoint operator S in a complex Hilbert space X, let $\varphi(v)$ be the proper lower semi-continuous convex function defined as

$$\varphi(v) := \begin{cases}(1/2)\|S^{1/2}v\|_X^2 & \text{if } v \in D(\varphi) := D(S^{1/2}),\\ +\infty & \text{otherwise,}\end{cases}$$

where $S^{1/2}$ is the square root of $S = \partial\varphi$. Then the equation in $(\mathbf{ACP})_\varepsilon$ is given by

$$(\text{ACPeq})_\varepsilon \quad D_t u_\varepsilon(t) + (\lambda + i\alpha)\partial\varphi(u_\varepsilon(t)) + (\kappa + i\beta)\partial\psi_\varepsilon(u_\varepsilon(t)) - \gamma u_\varepsilon(t) = 0$$

a.e. on $\mathbf{R}_+$, where $\psi \geq 0$ is another proper lower semi-continuous convex function on X.

Now we introduce five conditions on φ and ψ (or $S = \partial\varphi$ and $\partial\psi$).

(**A1**) $\{u \in X; \varphi(u) \leq c\}$ is compact in X $\forall\, c > 0$.

(**A2**) $\exists\, q \in [2, \infty);\ \psi(\zeta u) = |\zeta|^q \psi(u)\ \forall\, u \in D(\psi),\ \forall\, \zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$.

(**A3**) $\exists\, \theta \in [0, 1];\ \forall\, (\lambda, \kappa, \beta) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}\ \exists\, c_1 = c_1(\lambda, \kappa, \beta) > 0$ such that for $u \in D(S)$ and $\varepsilon > 0$,

$$|(Su, \partial\psi_\varepsilon(u))_X| \leq c_1 \psi(J_\varepsilon u)^\theta \varphi(u) + \frac{\lambda}{2\sqrt{\kappa^2 + \beta^2}} \|Su\|_X^2, \tag{4.1}$$

where $J_\varepsilon := (1 + \varepsilon\partial\psi)^{-1}$.

(**A4**) $D(S) \subset D(\partial\psi)$ and $\exists\, c_2 > 0$ such that

$$\|\partial\psi(u)\|_X \leq c_2(\|Su\|_X + \|u\|_X) \quad \forall\, u \in D(S).$$

(**A5**) $\forall\, (\lambda, \kappa, \beta) \in \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{R}\ \exists\, c_3 = c_3(\lambda, \kappa, \beta) > 0$ such that

$$\begin{aligned} &|(\partial\psi(u) - \partial\psi(v), u - v)_X| \\ &\leq c_3 \left[\frac{\psi(u) + \psi(v)}{2}\right]^\theta \|u - v\|_X^2 + \frac{2\lambda}{\sqrt{\kappa^2 + \beta^2}} \varphi(u - v) \end{aligned} \tag{4.2}$$

for $u, v \in D(\partial\psi) \cap D(\varphi)$. Here $\theta \in [0, 1]$ is the same constant as in (**A3**).

As a consequence of condition (**A1**) we can use compactness methods to prove the convergence of approximate solutions to $(\mathbf{ACP})_\varepsilon$. In (CGL) we assume that $\Omega \subset \mathbb{R}^n$ is **bounded** so that Rellich's theorem is available.

Now we can state two abstract theorems which lead us to Theorem 1.2. The first theorem is concerned with the existence of solutions to (**ACP**).

THEOREM 4.1 ([30]). *Let S be a nonnegative selfadjoint operator in X, ψ a proper lower semi-continuous convex function on X. Assume that conditions (**A1**)–(**A4**) are satisfied. Then for any initial value $u_0 \in D(\varphi) \cap D(\psi)$ there exists a strong solution u to (**ACP**) such that $u \in C^{0,1/2}([0, T]; X)$ $\forall\, T > 0$.*

In connection with (CGL) this theorem is interpreted as follows. To avoid the restriction on the coefficient $\kappa + i\beta$ of the nonlinear term $|u|^{q-2}u$ the size of the power $q-1$ is strictly restricted in Theorem 4.1 (cf. (1.4)). To see this $\theta \in [0,1]$ in condition (**A3**) is a key constant. In the process to verify (**A3**) θ is given by

$$\theta = f_N(q) := \frac{2(q-2)}{2q - N(q-2)}, \quad 2 \le q \le 2 + \frac{4}{N}; \tag{4.3}$$

note that $0 = f_N(2) \le f_N(q) \le f_N(2+4/N) = 1$. To the contrary, in [29, Theorem 4.1] there is a strong restriction on $\kappa + i\beta$, whereas the power is not restricted. Conditions (**A1**), (**A2**) are common in these theorems.

In the second theorem an additional condition (**A5**) guarantees the well-posedness of (**ACP**).

THEOREM 4.2 ([30]). *Let S and ψ be the same as in Theorem 4.1. Assume that conditions* (**A1**)*–*(**A5**) *are satisfied. Then for any $u_0 \in X = \overline{D(\varphi) \cap D(\psi)}$ there exists a* **unique** *strong solution u to* (**ACP**)*, with norm bound*

$$\|u(t)\|_X \le e^{\gamma t}\|u_0\|_X, \quad \forall\, t \ge 0, \tag{4.4}$$

such that $u \in C^{0,1/2}_{\mathrm{loc}}(\mathbf{R}_+; X)$. In addition, solutions depend continuously on initial data: there exists two constants $k_3 \in \mathbb{R}$, $k_4 \ge 0$ such that

$$\|u(t) - v(t)\|_X \le M(t, u_0, v_0)e^{k_3 t}\|u_0 - v_0\|_X \ \ \forall\, u_0, v_0 \in X. \tag{4.5}$$

Here $M(t, u_0, v_0) := \exp\bigl[k_4 e^{2\gamma_+ t}(\|u_0\|_X \vee \|v_0\|_X)^2\bigr]$.

REMARK 4.1 The coefficient on the right-hand side of (4.5) seems to be a little bit complicated. So it may be meaningful to describe a particular case of (4.5). Suppose that $\gamma \le 0$, and introduce the ball $B_L := \{v \in X; \|v\|_X \le L\}$. Setting $U(t)u_0 := u(t)$ and $\log M := k_4 L^2$, we have

$$\|U(t)u_0 - U(t)v_0\|_X \le Me^{k_3 t}\|u_0 - v_0\|_X \ \ \forall\, t \ge 0, \quad \forall\, u_0, v_0 \in B_L.$$

This implies by (4.4) that $\{U(t)\}$ forms a Lipschitz semigroup on the closed convex subset B_L in the sense of Kobayashi-Tanaka [19].

OUTLINE OF PROOFS. Let $u_0 \in D(\varphi) \cap D(\psi)$ and $\varepsilon > 0$. Then under condition (**A2**) there exists a unique strong solution u_ε to $(\mathbf{ACP})_\varepsilon$ such that for all $T > 0$, $u_\varepsilon \in C^{0,1/2}([0,T];X)$ ([29, Proposition 3.1(i)]). In addition we have

$$\|u_\varepsilon(t)\|_X \le e^{\gamma t}\|u_0\|_X \ \ \forall\, t \ge 0, \tag{4.6}$$

$$2\lambda\int_0^t \varphi(u_\varepsilon(s))\,ds + q\kappa\int_0^t \psi(J_\varepsilon u_\varepsilon(s))\,ds \le \frac{1}{2}e^{2\gamma_+ t}\|u_0\|_X^2, \quad \forall\, t \ge 0; \tag{4.7}$$

note that $\mathrm{Re}\,((\kappa+i\beta)\partial\psi_\varepsilon(u_\varepsilon),u_\varepsilon) \ge \kappa\mathrm{Re}\,(\partial\psi(J_\varepsilon u_\varepsilon),J_\varepsilon u_\varepsilon) = q\kappa\psi(J_\varepsilon u_\varepsilon)$ (see [29, Lemma 3.2]). Next, (4.7) yields under conditions (**A2**) and (**A3**) that

$$\varphi(u_\varepsilon(t)) \le e^{K(t,u_0)}\varphi(u_0), \quad \forall\, t \ge 0, \tag{4.8}$$

$$\int_0^t \|Su_\varepsilon(s)\|_X^2 ds \le \frac{2}{\lambda}e^{K(t,u_0)}\varphi(u_0), \quad \forall\, t \ge 0. \tag{4.9}$$

Here $K(t,u_0) := k_1 t + k_2 e^{2\gamma_+ t}\|u_0\|_X^2$, with

$$k_1 := 2\gamma_+ + (1-\theta)\,c_1\sqrt{\kappa^2+\beta^2}, \qquad k_2 := \theta\,c_1\sqrt{\kappa^2+\beta^2}/(2q\kappa).$$

Therefore we see from (4.6), (4.9), (**A4**) and $(\mathrm{ACPeq})_\varepsilon$ that

$$\{Su_\varepsilon\}, \{\partial\psi_\varepsilon(u_\varepsilon)\}, \{D_t u_\varepsilon\} \text{ are bounded in } L^2(0,T;X). \tag{4.10}$$

As a consequence $\{u_\varepsilon\}$ is equicontinuous on $[0,T]$:

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_X \le \|D_t u_\varepsilon\|_{L^2(0,T;X)}|t-s|^{1/2}, \quad \forall\, t,s \in [0,T]. \tag{4.11}$$

In view of condition (**A1**) (4.8) yields that for every $t \in [0,T]$ the family $\{u_\varepsilon(t)\}_{\varepsilon>0}$ is relatively compact in X. Therefore, applying Ascoli's theorem, we can find a function $u \in C([0,\infty);X)$ as the limit of a convergent subsequence $\{u_{\varepsilon_n}\}$ selected from $\{u_\varepsilon\}$ such that

$$u = \lim_{n\to\infty} u_{\varepsilon_n} \ \text{ in } \ C([0,T];X), \quad \forall\, T > 0, \tag{4.12}$$

with $u(0) = u_0 \in D(\varphi)\cap D(\psi)$. Letting $n \to \infty$ in (4.11) with ε replaced with ε_n, we have $u \in C^{0,1/2}([0,T];X)$ for any $T > 0$. Since $u_{\varepsilon_n} - J_{\varepsilon_n}u_{\varepsilon_n} = \varepsilon_n\partial\psi_{\varepsilon_n}(u_{\varepsilon_n})$, (4.10) yields that

$$u = \lim_{n\to\infty} J_{\varepsilon_n}u_{\varepsilon_n} \ \text{ in } \ L^2(0,T;X), \quad \forall\, T > 0. \tag{4.13}$$

Since S, D_t are weakly closed and $\partial\psi$ is demi-closed, we see from (4.12), (4.13) and (4.10) that $Su_{\varepsilon_n} \to Su$, $D_t u_{\varepsilon_n} \to D_t u$ and

$$\partial\psi_{\varepsilon_n}(u_{\varepsilon_n}) = \partial\psi(J_{\varepsilon_n}u_{\varepsilon_n}) \to \partial\psi(u) \quad \text{as } n \to \infty$$

weakly in $L^2(0,T;X)$. Therefore, we can conclude that u is a strong solution to (**ACP**) (for details see [29, Section 4]).

Here we want to mention two inequalities for strong solutions u to (**ACP**):

$$2\lambda \int_0^t \varphi(u(s))\,ds + q\kappa \int_0^t \psi(u(s))\,ds \le \frac{1}{2} e^{2\gamma_+ t}\|u_0\|_X^2, \ \ \forall\, t \ge 0, \quad (4.14)$$

$$t\varphi(u(t)) + \frac{\lambda}{2}\int_0^t s\|Su(s)\|_X^2\,ds \le \frac{1}{4\lambda} e^{K(t,\|u_0\|_X)+2\gamma_+ t}\|u_0\|_X^2, \ \ \forall\, t \ge 0, \quad (4.15)$$

which will be essential in the proof of Theorem 4.2. To prove (4.15) begin with (4.1) with $\partial\psi_\varepsilon$ and $J_\varepsilon u$, respectively, replaced with $\partial\psi$ and u (by letting $\varepsilon \downarrow 0$). (Then an error in the proof of (4.15) in [30] is easily corrected.)

The important role of condition (**A5**) in Theorem 4.2 is to derive (4.5) as a consequence of (4.14) and to guarantee the uniqueness of strong solutions to (**ACP**). The constants k_j ($j = 3, 4$) in (4.5) are given in almost the same form as for k_1 and k_2:

$$k_3 := \gamma + (1-\theta)\,c_3\sqrt{\kappa^2+\beta^2}, \qquad k_4 := \theta\,c_3\sqrt{\kappa^2+\beta^2}/(2q\kappa).$$

At the same time (4.5) enables us to extend the set of initial data from $D(\varphi)\cap D(\psi)$ to its closure $\overline{D(\varphi)\cap D(\psi)} = X$. The situation is completely similar to the inference in Theorem 3.2. In fact, (4.15) implies that $\|Su\|_{L^2(T^{-1},T;X)}$ is bounded for all $T > 1$.

Finally, we shall verify (4.1) and (4.2) when $S = \partial\varphi$, $B = \partial\psi$ are given by (2.1), (2.2), respectively. First we state a formula for the derivative of the function $J_\varepsilon u$ by which we can carry out integration by parts in the computation of $(Su, \partial\psi_\varepsilon(u))_{L^2}$.

LEMMA 4.1 *For $\varepsilon \in [0,\infty)$ and $x \in \Omega$ put*

$$u_\varepsilon(x) := (J_\varepsilon u)(x) = \begin{cases} (1+\varepsilon\partial\psi)^{-1}u(x), & \varepsilon > 0, \\ u(x), & \varepsilon = 0. \end{cases}$$

(a) ([29, Lemma 6.1 with $p = 2$]). *If $u \in H_0^1(\Omega)$, then $u_\varepsilon \in H_0^1(\Omega)$ (as a function of x), with*

$$\nabla_x u_\varepsilon = \begin{cases} \dfrac{1}{1+\varepsilon|u_\varepsilon|^{q-2}}\nabla_x u - \varepsilon\dfrac{q-2}{\mathrm{Jac}}|u_\varepsilon|^{q-4}u_\varepsilon \mathrm{Re}(\overline{u_\varepsilon}\,\nabla_x u), & \varepsilon > 0, \\ \nabla_x u, & \varepsilon = 0, \end{cases} \quad (4.16)$$

where $\mathrm{Jac} := (1+\varepsilon|u_\varepsilon|^{q-2})(1+\varepsilon(q-1)|u_\varepsilon|^{q-2})$ *is the Jacobian determinant.*

(b) ([31]). *If $u \in D(\partial\psi)$, then $u_\varepsilon \in C^1([0,E];L^2(\Omega))\ \forall\, E > 0$ (as a function of ε), with*

$$\frac{\partial u_\varepsilon}{\partial\varepsilon} = \begin{cases} -\dfrac{1}{1+\varepsilon(q-1)|u_\varepsilon|^{q-2}}\partial\psi_\varepsilon(u), & \varepsilon > 0, \\ -\partial\psi(u), & \varepsilon = 0. \end{cases}$$

For a proof of (a) put $u_\varepsilon(x) = v_\varepsilon(x) + iw_\varepsilon(x)$. Then, applying the implicit (or inverse) function theorem to the simultaneous equation derived from the real and imaginary parts of $u_\varepsilon + \varepsilon|u_\varepsilon|^{q-2}u_\varepsilon = u \in H_0^1(\Omega) \cap C^1(\overline{\Omega})$, we can obtain the formulas for $\nabla_x v_\varepsilon$ and $\nabla_x w_\varepsilon$. Therefore, Jac appears when we solve the simultaneous equation for $\nabla_x v_\varepsilon$ and $\nabla_x w_\varepsilon$. In the second step the usual approximating procedure yields (4.16).

Verification of (4.1). Put $u_\varepsilon = J_\varepsilon u$ as in Lemma 4.1. Then in view of the form of Yosida approximation $\partial\psi_\varepsilon(u) = \varepsilon^{-1}(u - u_\varepsilon)$ we see from (4.16) that $(Su, \partial\psi_\varepsilon(u))_{L^2}$ is written as

$$\int_\Omega \frac{|u_\varepsilon|^{q-2}}{1+\varepsilon|u_\varepsilon|^{q-2}}|\nabla u|^2\,dx + \int_\Omega \frac{q-2}{\mathrm{Jac}}|u_\varepsilon|^{q-4}(\overline{u_\varepsilon}\nabla u)\cdot \mathrm{Re}\,(\overline{u_\varepsilon}\nabla u)\,dx, \tag{4.17}$$

where $\nabla := \nabla_x$. Using Hölder's inequality, we have, for $u \in H_0^1(\Omega) \cap H^2(\Omega)$,

$$(q-1)^{-1}|(Su, \partial\psi_\varepsilon(u))_{L^2}| \le \int_\Omega |u_\varepsilon|^{q-2}|\nabla u|^2\,dx \le \|u_\varepsilon\|_{L^q}^{q-2}\|\nabla u\|_{L^q}^2. \tag{4.18}$$

In the next step we need the inequality

$$\|\nabla u\|_{L^q} \le C\|\nabla u\|_{L^2}^{1-a}\|\Delta u\|_{L^2}^a, \quad 0 \le a := \frac{N}{2} - \frac{N}{q} \le 1. \tag{4.19}$$

This is a consequence of the well known Gagliardo-Nirenberg interpolation inequality (see (4.22) below). Let θ be as defined by (4.3). Then, applying (4.19) and Young's inequality, we can show that for any $\eta > 0$ there exists $C(\eta) > 0$ such that (4.1) holds:

$$|(Su, \partial\psi_\varepsilon(u))_{L^2}| \le C(\eta)\psi(u_\varepsilon)^\theta\|S^{1/2}u\|_{L^2}^2 + \eta\|Su\|_{L^2}^2, \;\; u_\varepsilon = J_\varepsilon u.$$

Thus we can explain the restriction on q:

$$\theta = \frac{q-2}{(1-a)q} \le 1 \iff a = \frac{N}{2} - \frac{N}{q} \le \frac{2}{q} \iff q \le 2 + \frac{4}{N}.$$

REMARK 4.2 It is worth noticing that (3.6) follows from (4.17). The proof is fairly simple (see [29, Lemma 6.2 with $p = 2$]). In fact, setting

$$I_1(u) := \int_\Omega \frac{|u_\varepsilon|^{q-2}}{1+\varepsilon|u_\varepsilon|^{q-2}}|\nabla u|^2\,dx,$$

$$I_2(u) := \int_\Omega \frac{|u_\varepsilon|^{q-4}}{\mathrm{Jac}}(\overline{u_\varepsilon}\nabla u)\cdot \mathrm{Re}\,(\overline{u_\varepsilon}\nabla u)\,dx,$$

we see from (4.17) that the real and imaginary parts of $(Su, \partial\psi_\varepsilon(u))_{L^2}$ have similar expressions:

$$\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_{L^2} = I_1(u) + (q-2)\mathrm{Re}\, I_2(u) \geq 0, \tag{4.20}$$

$$\mathrm{Im}\,(Su, \partial\psi_\varepsilon(u))_{L^2} = (q-2)\mathrm{Im}\, I_2(u). \tag{4.21}$$

Applying the Cauchy-Schwarz inequality to $I_2(u)$, we have

$$|I_2(u)|^2 \leq I_1(u)\mathrm{Re}\, I_2(u)$$

which can be combined with (4.20) and (4.21) as follows:

$$\begin{aligned}
&(q-2)^{-2}|\mathrm{Im}\,(Su, \partial\psi_\varepsilon(u))_{L^2}|^2 = |\mathrm{Im}\, I_2(u)|^2 = |I_2(u)|^2 - |\mathrm{Re}\, I_2(u)|^2 \\
&\leq I_1(u)\mathrm{Re}\, I_2(u) - [\mathrm{Re}\, I_2(u)]^2 \\
&= [\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_{L^2} - (q-2)\mathrm{Re}\, I_2(u)]\mathrm{Re}\, I_2(u) - [\mathrm{Re}\, I_2(u)]^2 \\
&= \frac{1}{\sqrt{q-1}}\,\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_{L^2}\sqrt{q-1}\,[\mathrm{Re}\, I_2(u)] - (q-1)[\mathrm{Re}\, I_2(u)]^2 \\
&\leq \frac{1}{4\,(q-1)}[\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_{L^2}]^2.
\end{aligned}$$

In the last step we have used the inequality $ab = 2(a/2)b \leq (a/2)^2 + b^2$. Since $\mathrm{Re}\,(Su, \partial\psi_\varepsilon(u))_{L^2} \geq 0$ as pointed out in (4.20), we obtain (3.6) with $X = L^2$ and $\tan\omega_q = c_q$.

Verification of (4.2). First we prepare an inequality asserting the local Lipschitz continuity of the mapping $z \mapsto |z|^{q-2}z = \partial(|z|^q/q)$:

$$\begin{aligned}
\frac{\big||z|^{q-2}z - |w|^{q-2}w\big|}{(q-1)|z-w|} &\leq \int_0^1 |sz + (1-s)w|^{q-2}\, ds \\
&\leq \Big(\frac{1}{2}|z|^q + \frac{1}{2}|w|^q\Big)^{(q-2)/q} \quad \forall\, z, w \in \mathbf{C}\ (z \neq w);
\end{aligned}$$

in the proof we have used the convexity of the function $z \mapsto |z|^q$, which is a simple consequence of Hölder's inequality $(1/q + 1/q' = 1)$:

$$\begin{aligned}
&|sz + (1-s)w|^q \\
&\leq (s|z| + (1-s)|w|)^q = \Big(s^{1/q}|z| \cdot s^{1/q'} + (1-s)^{1/q}|w| \cdot (1-s)^{1/q'}\Big)^q \\
&\leq (s|z|^q + (1-s)|w|^q)^{q/q}(s + (1-s))^{q/q'} = s|z|^q + (1-s)|w|^q.
\end{aligned}$$

Then we can prove the inequality for $u, v \in L^{2(q-1)}(\Omega) \cap H_0^1(\Omega) \subset L^q(\Omega)$:

$$\begin{aligned}
&(q-1)^{-1}\big|(|u|^{q-2}u - |v|^{q-2}v, u-v)_{L^2}\big| \\
&\le \int_\Omega |u(x)-v(x)|^2\Big(\frac{1}{2}|u(x)|^q + \frac{1}{2}|v(x)|^q\Big)^{(q-2)/q}\,dx \\
&\le \Big(\frac{1}{2}\|u\|_{L^q}^q + \frac{1}{2}\|v\|_{L^q}^q\Big)^{(q-2)/q}\|u-v\|_{L^q}^2 \\
&= \Big(\frac{q}{2}\psi(u) + \frac{q}{2}\psi(v)\Big)^{(q-2)/q}\|u-v\|_{L^q}^2.
\end{aligned}$$

This is nothing but (4.18) (in Verification of (4.1)) with $\|u_\varepsilon\|_{L^q} = (q\psi(u_\varepsilon))^{1/q}$ and $\|\nabla u\|_{L^q}$ replaced with $\big[(q/2)(\psi(u)+\psi(v))\big]^{1/q}$ and $\|u-v\|_{L^q}$, respectively. From now on we can compute in the same way as in the proof of (4.1) to conclude (4.2); in this case we can apply the Gagliardo-Nirenberg inequality itself (see Tanabe [33, Section 3.4] or Giga-Giga [11, Section 6.1]):

$$\|w\|_{L^q} \le C\|w\|_{L^2}^{1-a}\|\nabla w\|_{L^2}^a, \quad 0 \le a := \frac{N}{2} - \frac{N}{q} \le 1. \tag{4.22}$$

Therefore, the "θ" coincides with that in (4.1). This result seems to be parallel to the fact that (3.2) and (3.4) in Theorem 3.1 hold for the same $\tan\omega_q$.

5 Development in the future

As a concluding remark we want to mention the possibility to unify Theorems 3.2 and 4.2 (see Remark 1.1). Now we introduce the following seven conditions on S and ψ:

(B1) $\exists\, q \in [2,\infty)$ such that $\psi(\zeta u) = |\zeta|^q\psi(u)$ for $u \in D(\psi)$ and $\zeta \in \mathbb{C}$ with $\operatorname{Re}\zeta > 0$.

(B2) $\partial\psi$ is sectorially-valued: $\exists\, c_q > 0$ such that

$$|\operatorname{Im}(\partial\psi(u)-\partial\psi(v), u-v)| \le c_q \operatorname{Re}(\partial\psi(u)-\partial\psi(v), u-v) \;\; \forall\, u, v \in D(\partial\psi).$$

(B3) For the same constant c_q as in (**B2**),

$$|\operatorname{Im}(Su, \partial\psi_\varepsilon(u))| \le c_q \operatorname{Re}(Su, \partial\psi_\varepsilon(u)) \;\; \forall\, u \in D(S).$$

(B4) $D(S) \subset D(\partial\psi)$ and $\exists\, C_1 > 0$ such that $\|\partial\psi(u)\| \le C_1(\|u\| + \|Su\|)$ for $u \in D(S)$.

(**B5**) $\exists\, \theta \in [0,1];\ \forall\, \eta > 0\ \exists\, C_2 = C_2(\eta) > 0$ such that for $u, v \in D(\varphi) \cap D(\psi)$ and $\varepsilon > 0$,

$$|\mathrm{Im}(\partial\psi_\varepsilon(u) - \partial\psi_\varepsilon(v), u - v)| \le \eta\varphi(u - v) + C_2 \Big[\frac{\psi(u) + \psi(v)}{2}\Big]^\theta \|u - v\|^2.$$

(**B6**) $\forall\, \eta > 0\ \exists\, C_3 = C_3(\eta) > 0$ such that for $u \in D(S)$ and $\varepsilon > 0$,

$$|\mathrm{Im}(Su, \partial\psi_\varepsilon(u))| \le \eta\|Su\|^2 + C_3\psi(u)^\theta\varphi(u),$$

where $\theta \in [0,1]$ is the same constant as in (**B5**).

(**B7**) $\exists\, C_4 > 0$ such that for $u, v \in D(\partial\psi)$ and $\nu, \mu > 0$,

$$|\mathrm{Im}(\partial\psi_\nu(u) - \partial\psi_\mu(u), v)| \le C_4|\nu - \mu|\big(\sigma\|\partial\psi(u)\|^2 + \tau\|\partial\psi(v)\|^2\big),$$

where σ, $\tau > 0$ are constants satisfying $\sigma + \tau = 1$.

Using these conditions (partially or as a whole) we can assert

THEOREM 5.1 ([31]). *Let $\lambda, \kappa \in \mathbb{R}_+$ and $\alpha, \beta, \gamma \in \mathbb{R}$.*

(**I**) *(Accretive nonlinearity) Assume that $|\beta|/\kappa \in [0, c_q^{-1}]$ and conditions* (**B1**)–(**B3**) *are satisfied. Then for any initial value $u_0 \in \overline{D(\varphi) \cap D(\psi)}$ there exists a unique strong solution $u \in C([0,\infty); X)$ to (**ACP**) such that*

(a) $u \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}_+; X)$, *with* $\|u(t)\| \le e^{\gamma t}\|u_0\|\ \ \forall\, t \ge 0$;

(b) $Su,\ \partial\psi(u),\ D_t u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+; X)$;

(c) $\varphi(u),\ \psi(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+)$.

*Furthermore, let v be the unique strong solution to (**ACP**) with $v(0) = v_0 \in X$. Then*

$$\|u(t) - v(t)\| \le e^{\gamma t}\|u_0 - v_0\| \quad \forall\, t \ge 0. \tag{5.1}$$

(**II**) *(Nonaccretive nonlinearity) Assume that $|\beta|/\kappa \in (c_q^{-1}, \infty)$ and the seven conditions* (**B1**)–(**B7**) *are satisfied. Then for any $u_0 \in X = \overline{D(S)}$ there exists a unique strong solution $u \in C([0,\infty); X)$ to (**ACP**). Also, u has property* (c) *in part* (**I**) *and*

(a)′ $u \in C^{0,1/2}_{\mathrm{loc}}(\mathbb{R}_+; X)$, *with* $\|u(t)\| \le e^{\gamma t}\|u_0\| \quad \forall\, t \ge 0$;

(b)′ $Su,\ \partial\psi(u),\ D_t u \in L^2_{\mathrm{loc}}(\mathbb{R}_+; X)$.

*Furthermore, let v be the unique strong solution to (**ACP**) with $v(0) = v_0 \in X$. Then*

$$\|u(t) - v(t)\| \le e^{K_1 t + K_2 e^{2\gamma_+ t}(\|u_0\| \vee \|v_0\|)^2}\|u_0 - v_0\| \quad \forall\, t \ge 0, \tag{5.2}$$

where the constants K_1 and K_2 depend on $|\beta| - c_q^{-1}\kappa > 0$:

$$K_1 := \gamma + (1-\theta)(|\beta| - c_q^{-1}\kappa)C_2, \quad K_2 := \theta\,(|\beta| - c_q^{-1}\kappa)C_2/(2q\kappa). \tag{5.3}$$

Part (**I**) is nothing but Theorem 3.2 (in view of Remark 3.1), while Part (**II**) generalizes Theorem 4.2 to the effect that the compactness of the level sets for φ is replaced with the new condition (**B7**) which guarantees, together with condition (**B5**), the convergence of the family of solutions to $(\mathbf{ACP})_\varepsilon$; the verification of (**B7**) is based on Lemma 4.1 (b). Therefore the boundedness of Ω is not required when we apply Theorem 5.1 Part (**II**) to problem (CGL).

REMARK 5.1 It is observed from (5.3) that $K_1 \to \gamma$ and $K_2 \to 0$ as $|\beta| \to c_q^{-1}\kappa$, that is, (5.2) coincides with (5.1) in this limit.

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